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RESEARCH ARTICLE

ANALYTICAL SOLUTION OF RECTANGULAR PLATE DUE TO CONVECTIVE HEAT TRANSFER BY DISSIPATION

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ABSTRACT

The aim of this study was to investigate temperature distribution and thermal stress analysis of isotropic rectangular plate due to convection by dissipation. The convection heat is applied in the form of partial distribution which is represented by Heaviside unit step function. The finite element formulation has been developed to solve governing heat conduction equation which results the rate of heat transfer and temperature distribution within rectangular plate. Also the classical theory of thermo elasticity has been used to develop the finite element formulation for thermo elastic stress analysis. The results of temperature and stress have been computed numerically, illustrated graphically and interpreted technically.

INTRODUCTION

As a result of the increased usage of industrial and construction material the interest in isotropic thermal stress problem has grown considerably. The finite element method has become a powerful tool for the numerical solution of a wide range of engineering problems. The basic idea of finite element method was developed by Turner *et al.* (1956) to obtain the approximate solution of a complicated problem by replacing it into a simpler problem rather than obtaining the exact solution. In the finite element method it will often be possible to improve or refine the approximate solution by spending more computational effort. Dechaumphai *et al.* (1996) used finite element analysis procedure for predicting temperature and thermal stresses of heated products. David *et al.* (2008) calculated convection heat transfer, predicted convection cooling and flow in electric machines. Flow network analysis is used to study the ventilation inside the machine. Dhawan *et al.* (2009) observed temperature decay in an aluminum plate using Galerkin finite element method for 2D transient heat conduction equation. A comparative study has been made taking different combinations of meshes and numerical schemes. Vanam *et al.* (2012), performed static analysis of an isotropic rectangular plate with various boundary conditions and various types of load applications with four noded quadrilateral element using finite element analysis (FEA). The results obtained from Numerical calculation, MATLAB and software ANSYS are compared. It is helpful for obtaining the results not only at node points but also the entire surface of the rectangular plate.

This paper deals with the realistic problem of the thermal stresses of isotropic rectangular plate subjected to the parallel edges $x = 0$ and $x = a$ of rectangular plate are thermally insulated. The edge $y = b$ is subjected to convective heat transfer whereas heat dissipates by convection from the lower boundary surface $y = 0$. The convection heat is applied in the form of partial distribution which is represented by Heaviside unit step function. The initial temperature of the rectangular plate is kept at constant temperature T_i. The governing heat conduction equation has been solved by using finite element method (FEM). The temperature distribution, thermal stress analysis and the corresponding finite element formulation are presented. The Matlab programming is used to evaluate the temperature at different nodes and thermal stresses along X, Y and shear stress in resultant direction in the rectangular plate. The results presented in this paper have better accuracy since numerical calculations have been performed for discretization of the large number of elements in rectangular plate.

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To the best of our knowledge no one has developed finite element model for heat transfer and thermal stress analysis due to convection heat supply in the form of partial distribution and convection by dissipation in a rectangular plate. This is a new and novel contribution to the field. The results presented here will be useful in engineering problems particularly in determination of the state of stress in rectangular plate subjected to convection by dissipation.

FORMULATION OF THE PROBLEM

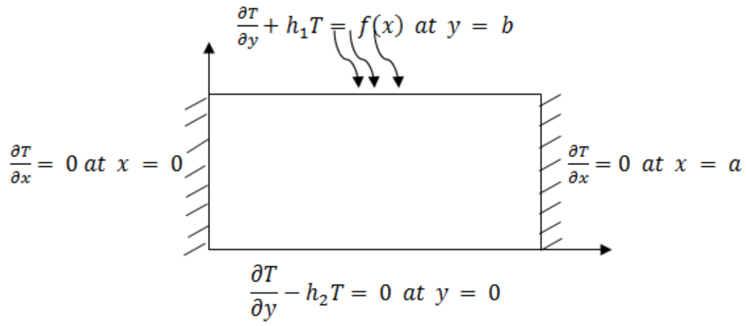

Figure 1. Schematic 2D rectangular plate subjected to convection

Figure 1 shows the schematic sketch of thin rectangular plate with given boundary condition. The governing two dimensional transient heat conduction differential equations occupying the space $\{D: 0 \le x \le a, 0 \le y \le b\}$ is given as

$$k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) = c\rho \frac{\partial T}{\partial t} \qquad (1)$$

where ρ is the density of the solid, c is the specific heat and k is the thermal conductivity of material used in a rectangular plate. The rectangular plate is subjected to the convection heat transfer at the edge $y = b$, convection due to dissipation takes place from the edge $y = 0$ and parallel edges $x = 0,\ x = a$ are thermally insulated. The prescribed boundary conditions are as

$$\frac{\partial T}{\partial x} = 0 \quad at \quad x = a,\ 0 \le y \le b \qquad (2)$$

$$\frac{\partial T}{\partial x} = 0 \quad at \quad x = 0,\ \ 0 \le y \le b \qquad (3)$$

$$\frac{\partial T}{\partial y} + h_1 T = f(x) \quad at \quad y = b,\ 0 \le x \le a \qquad (4)$$

$$\frac{\partial T}{\partial y} - h_2 T = 0 \quad at \quad y = 0,\ 0 \le x \le a \qquad (5)$$

$$T(x,\ y,\ t) = T_i \quad at \quad t = 0,\ 0 \le x \le a,\ 0 \le y \le b \qquad (6)$$

where $f(x)$ represents partially distributed heat supply which can be represented in terms of Heaviside unit step function $\{H(x)\}$ and $\{H(x - x_0)\}$. Hence function is defined as

$$f(x) = T_o\{H(x) - H(x - x_0)\}$$

where $H(x) = \begin{Bmatrix} 1, & x > 0 \\ 0, & x \le 0 \end{Bmatrix}$ and $H(x - x_0) = \begin{Bmatrix} 1, & x > x_0 \\ 0, & x \le x_0 \end{Bmatrix}$

and the point of partial distribution heat supply is $x_0 = \frac{a}{2}$.

MATERIALS AND METHODS

Galerkin's finite element approach

Galerkin's finite element is applied to reduce differential equation into system of algebraic equation as in David Hutton (2005). Finite element formulation can be developed for two dimensional isotropic rectangular plate, assuming two dimensional element having *M* nodes such that the temperature distribution in the element is described by

$$T(x,y,t) = \sum_{i=1}^{M} N_i(x,y) T_i(t) = [N]^T\{T\} \qquad (7)$$

where $N_i(x,y)$ is the interpolation or shape function associated with nodal temperature T_i, $[N]$ is the row matrix of interpolation functions and $\{T\}$ is the column or vector matrix of nodal temperatures. Applying Galerkin's finite element method the residual equations corresponding to equation *(1)* for all $i = 1,2,\ldots\ldots M$ is given by

$$\iint_A N_i(x,y)\left[\left(k\frac{\partial^2 T}{\partial x^2}+k\frac{\partial^2 T}{\partial y^2}\right)-\rho c\frac{\partial T}{\partial t}\right]dA = 0$$

Integrating by parts the first two terms of the above equation reduces to

$$\iint_A \left[k\frac{\partial N_i(x,y)}{\partial x}\frac{\partial T}{\partial x}+k\frac{\partial N_i(x,y)}{\partial y}\frac{\partial T}{\partial y}+\rho c N_i(x,y)\frac{\partial T}{\partial t}\right]dA-\int_0^a\left[kN_i(x,y)\frac{\partial T}{\partial y}\right]_{y=0}^{b}dx-\int_0^b\left[kN_i(x,y)\frac{\partial T}{\partial x}\right]_{x=0}^{a}dy = 0$$

Applying the boundary conditions *(2)*, *(3)*, *(4)* and *(5)* the above equation becomes

$$k\iint_A \frac{\partial N_i(x,y)}{\partial x}\frac{\partial T}{\partial x}dA + k\iint_A \frac{\partial N_i(x,y)}{\partial y}\frac{\partial T}{\partial y}dA + \rho c\iint_A N_i(x,y)\frac{\partial T}{\partial t}dA$$
$$= \int_0^a \{kN_i(x,y)[f(x)-h_1T]\}_{y=b}\,dx - \int_0^a [kN_i(x,y)h_2T]_{y=0}\,dx$$

Using equation *(7)* and $\frac{\partial T}{\partial t} = \{\dot{T}\}$ the above equation reduces to

$$k\iint_A \frac{\partial[N]}{\partial x}\frac{\partial[N]^T}{\partial x}\{T\}dA + k\iint_A \frac{\partial[N]}{\partial y}\frac{\partial[N]^T}{\partial y}\{T\}dA + \rho c\iint_A [N][N]^T\{\dot{T}\}dA = k\int_{x=0}^{a}[N]_{y=b}f(x)dx - kh_1\int_0^a[[N][N]^T]_{y=b}\{T\}\,dx - kh_2\int_0^a\left[[N][N]^T\right]_{y=0}\{T\}\,dx$$

Substituting the value of *f(x)* the above equation becomes

$$k\iint_A \frac{\partial[N]}{\partial x}\frac{\partial[N]^T}{\partial x}\{T\}dxdy + k\iint_A \frac{\partial[N]}{\partial y}\frac{\partial[N]^T}{\partial y}\{T\}dxdy + kh_1\int_0^a[[N][N]^T]_{y=b}\{T\}\,dx + kh_2\int_0^a\left[[N][N]^T\right]_{y=0}\{T\}\,dx$$
$$+\rho c\iint_A [N][N]^T\{\dot{T}\}dxdy = kT_o\int_{x=0}^{a}[N]_{y=b}\{H(x)-H(x-x_0)\}dx$$

which is of the form

$$\{[K_c]+[K_{h1}]+[K_{h2}]\}\{T\}+[C]\{\dot{T}\}=\{f_g\}$$

$$[K]\{T\}+[C]\{\dot{T}\}=\{f_g\}$$

where the characteristic or conductance matrix $[K_c]$ is

$$[K_c] \triangleq k\iint_A\left[\frac{\partial[N]}{\partial x}\frac{\partial[N]^T}{\partial x}+\frac{\partial[N]}{\partial y}\frac{\partial[N]^T}{\partial y}\right]dxdy \qquad (8)$$

the characteristic matrix due to convection heat transfer $[K_{h1}]$ is

$$[K_{h1}] \triangleq kh_1\int_0^a[[N][N]^T]_{y=b}\,dx \qquad (9)$$

the characteristic matrix with dissipation due to convection $[K_{h2}]$ is

$$[K_{h2}] \triangleq kh_2\int_0^a\left[[N][N]^T\right]_{y=0}dx \qquad (10)$$

the capacitance matrix $[C]$ is

$$[C] \triangleq \rho c\iint_A[N][N]^T dA \qquad (11)$$

and the gradient or load matrix $[f_g]$ is

$$\{f_g\} \triangleq kT_o\int_{x=0}^{a}[N]_{y=b}\{H(x)-H(x-x_0)\}dx \qquad (12)$$

Element formulation

Formulate the three node triangular element shape function [1, 7]. The interpolation function for three nodes triangular element as

$$N_1(x,y) = \frac{1}{2A}[(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y] = \frac{A_1}{A}$$

$$N_2(x,y) = \frac{1}{2A}[(x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y] = \frac{A_2}{A}$$

$$N_3(x,y) = \frac{1}{2A}[(x_1y_2 - x_2y_1) + (y_1 - y_2)x + (x_2 - x_1)y] = \frac{A_3}{A}$$

where the area of the triangular element is given as

$$A = \frac{1}{2}\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}$$

NUMERICAL CALCULATION AND RESULTS

The aluminum rectangular plate occupying as length $(a) = 2\,ft$, breath $(b) = 1\,ft$ and thickness $(c_t) = 0.1\,ft$ with initial temperature$(T_i) = 104\ (F^0)$ and temperature applied on the surface $y = b\ \ (T_o) = 212\ (F^o)$. Parameter values used in this problem are density of the solid$(\rho) = 169\ (lb/ft^3)$, thermal conductivity of the material$(k) = 117\ (BTU/(hr - ft - F^0))$, specific heat$(c) = 0.208\ (BTU/(lb - F^0))$, linear coefficient of thermal expansion $(\alpha) = 12.84\,x10^{-6}(1/F^0)$, Young's modulus of elasticity of the material plate $(E) = 69\ (GPa)$, poisson's ratio $(v) = 0.35$, coefficient of convection at $(y = b)$ surface $(h_1) = 10$, coefficient of convection at$(y = 0)$ surface $(h_2) = 10\ a$nd point of partial heat supply $(x_0) = 1\ (ft)$.

Temperature Distribution

The rectangular plate of dimension $a = 2$ and $b = 1$ is discretized into *66* numbers of nodes *(M)* and *100* numbers of triangular elements *(NE)* shown in the figure *2*.

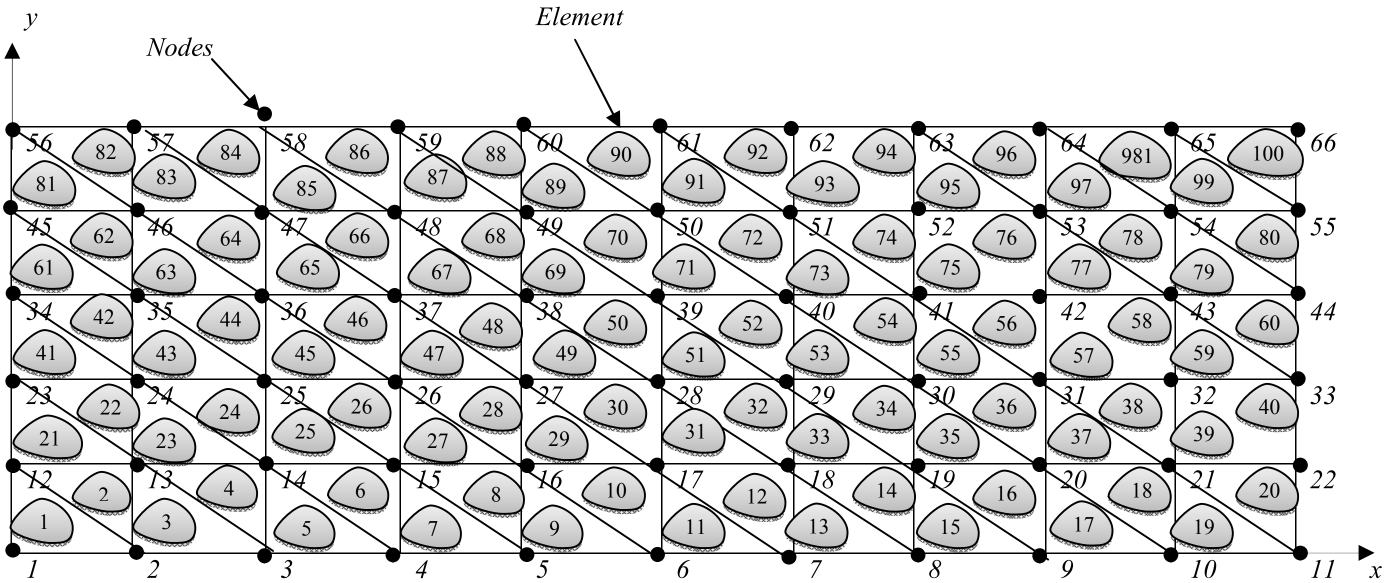

Figure 2. Geometry showing node and element position in 2D rectangular plate

Solving equations 8, 9, 10, 11, 12 we get the elemental matrix then assembled all the 100 elements and formed the final equation as given below

$$[K]\{T\} + [C]\{\dot{T}\} = \{f_g\}$$

Applying the Finite difference method and substituting $\{\dot{T}\} = \left\{\frac{T(t+\Delta t)-T(t)}{\Delta t}\right\}$, the above equation reduces to

$$[K]\{T\} + [C]\left\{\frac{T(t+\Delta t) - T(t)}{\Delta t}\right\} = \{f_g\}$$

which is simplified as under

$$T(t+\Delta t) = [C]^{-1}\{f_g\}\Delta t - [C]^{-1}[K]\{T\}\Delta t + \{T\}$$

Solving above equation by using Matlab programming one obtains the temperatures on all the nodes at different time. The numerical values of temperature distribution along *X* and *Y* axes in the rectangular plate at time *t=5* sec due to convection is shown in the Figures 3(a) and 3(b), respectively.

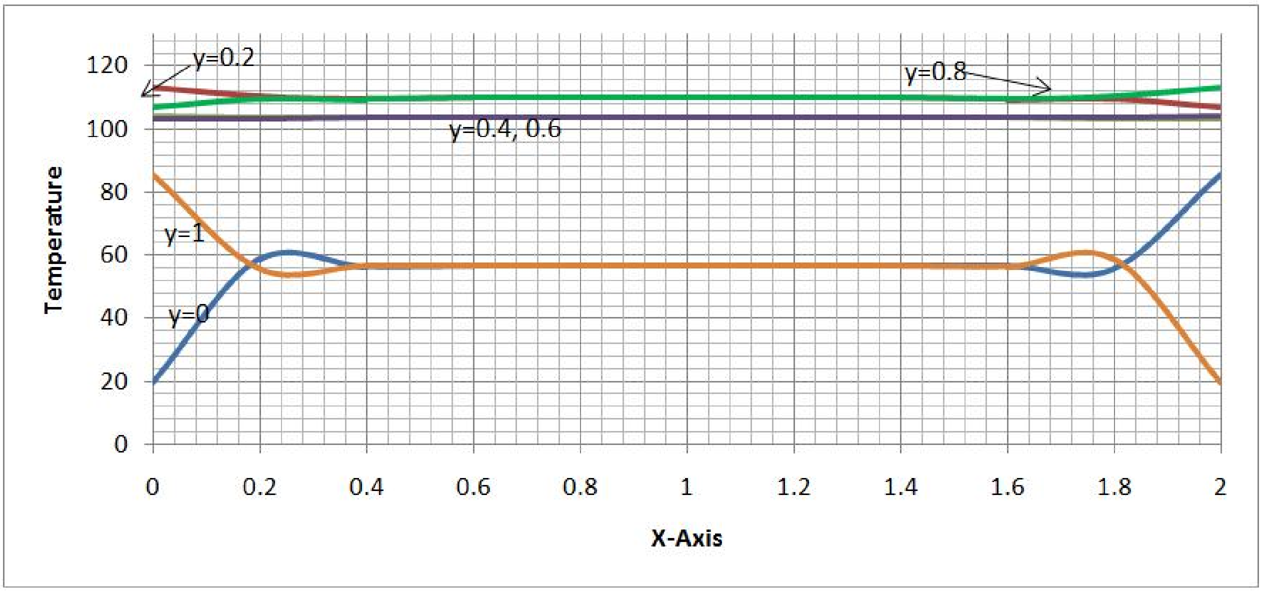

Figure 3a. Temperature distribution along *X* direction

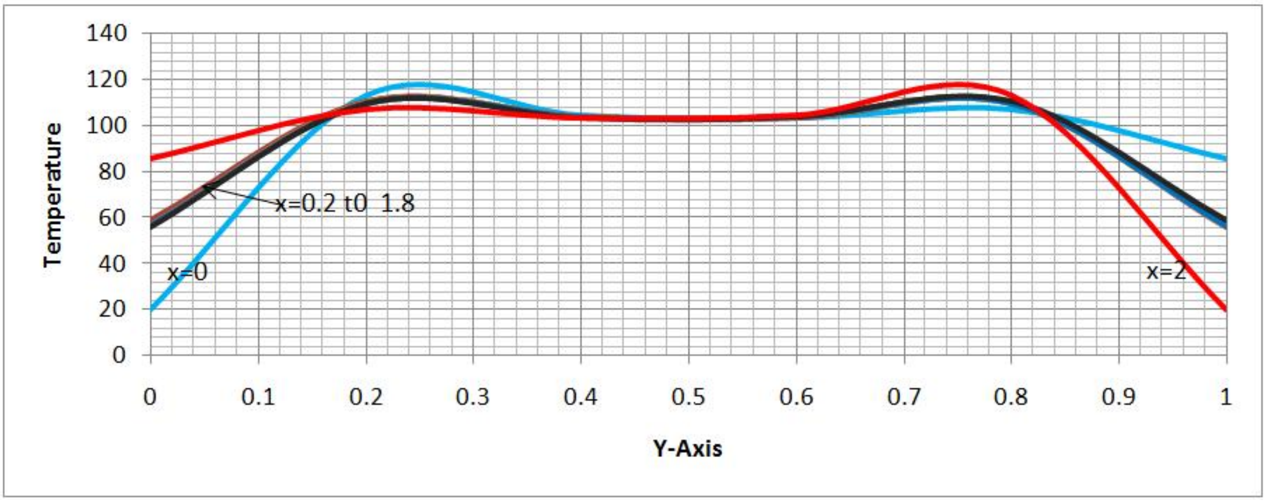

Figure 3b. Temperature distribution along *Y* direction

Displacement Analysis

The displacement components of node *j* are taken as q_{2j-1} in the *X* direction and q_{2j} in the *Y* direction. It denotes the global displacement vector as $Q = [q_{1,}q_2, q_3, \ldots \ldots \ldots \ldots . q_{132}]^T$ as shown in the figure *4*.

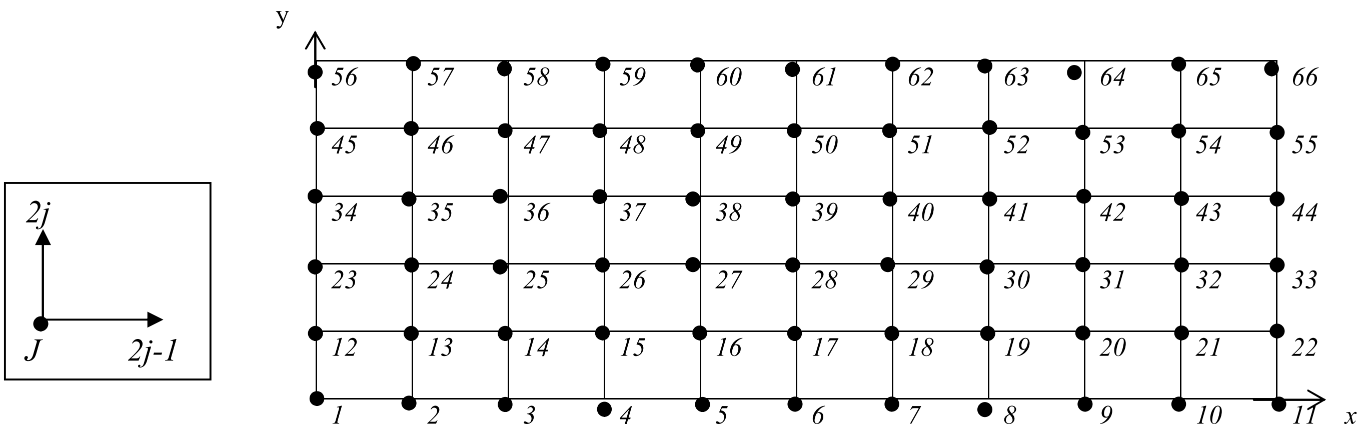

Figure 4. Geometry showing nodal displacement

Obtaining the displacements in the rectangular plate, the element stiffness matrix is defined as

$$[K^e] = c_t A_e B^T DB$$

As the distribution of the change in Temperature $\Delta T(x, y) = \Delta T = T_{ij} = T_i - T_j$ is known, the strain due to this change in temperature can be treated as an initial strain ϵ_0 and the element temperature load matrix θ^e can be represented as

$$[\epsilon_0] = [\alpha\Delta t\ , \alpha\Delta t\ , 0]^T \text{ and } [\theta^e] = c_t A_e B^T D \epsilon_0$$

Where

$$[D] = \frac{E}{(1-v^2)}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix}$$

$$[\mathrm{B}] = \frac{1}{\det J}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix} \text{ and } [J] = \begin{bmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{bmatrix}$$

using the notation, $x_{ij} \triangleq x_i - x_j$ and $y_{ij} \triangleq y_i - y_j$.

Assembling all the elemental stiffness and temperature load matrix, the final stiffness matrix K and temperature load matrix θ are formed and expressed as

$$KQ = \theta$$

The above equation is solved by using Gaussian elimination method to yield the displacement Q at the nodes along X and Y is shown in the Figures 5(a) and 5(b), respectively.

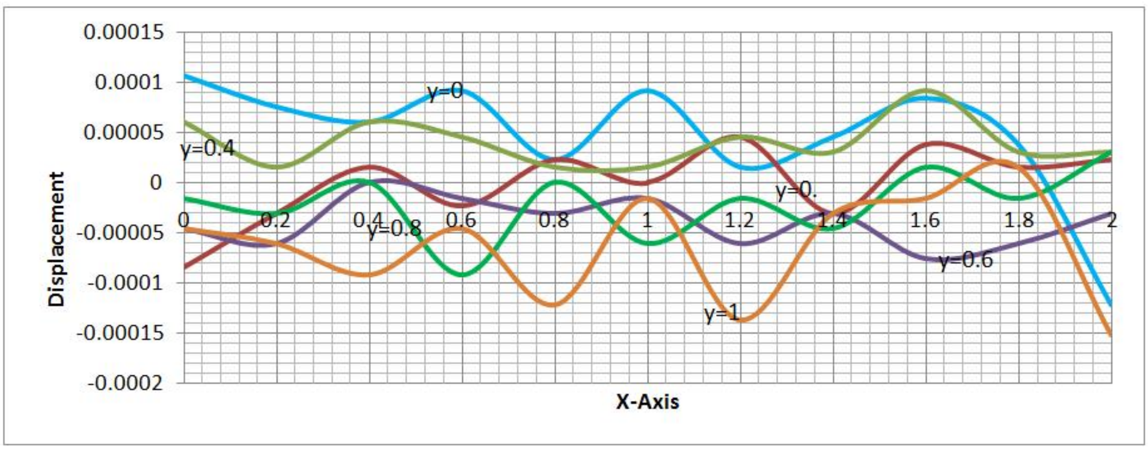

Figure 5a. Displacement along *X* direction

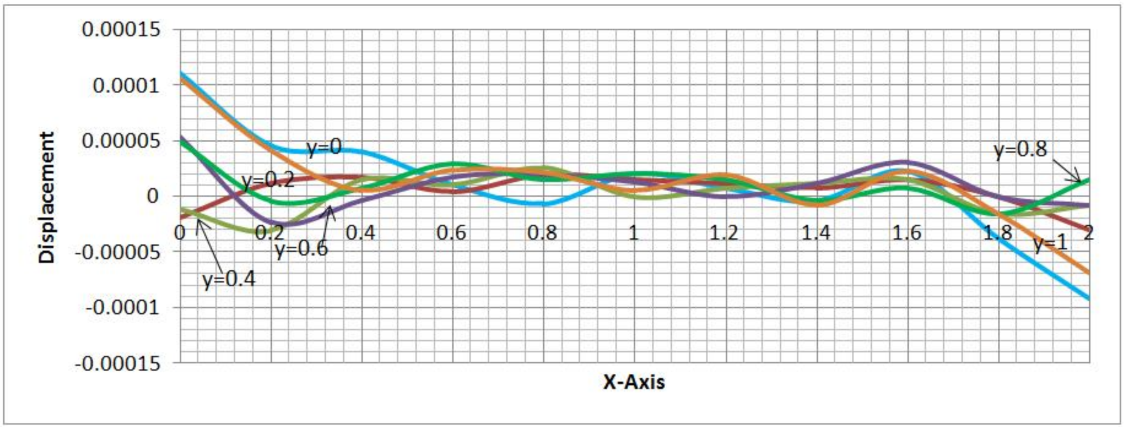

Figure 5b. Displacement along *Y* direction

Stress Analysis

Following the approach as in Robert cook *et al*. (2000), let u and v be the displacements along X and Y directions. The displacement inside the element are written using nodal values of the unknown displacement field as

$u = N_1q_1 + N_2q_3 + N_3q_5$ and $v = N_1q_2 + N_2q_4 + N_3q_6$

The shape function represented by area (A) co-ordinates are

$N_1 = \frac{A_1}{A}$, $N_2 = \frac{A_2}{A}$ and $N_3 = \frac{A_3}{A}$

Using the Strain-displacement relation [1] one gets

$$\in = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \end{Bmatrix} = \frac{1}{detJ}\begin{Bmatrix} y_{23}q_1 + y_{31}q_3 + y_{12}q_5 \\ x_{32}q_2 + y_{13}q_4 + y_{21}q_6 \\ x_{32}q_2 + y_{13}q_4 + y_{21}q_6 + y_{23}q_1 + y_{31}q_3 + y_{12}q_5 \end{Bmatrix}$$

This elemental equation can be written in matrix form as $\in^e = B^e q^e$.The Stress-strain relation is

$\sigma^e = E(\epsilon^e - \epsilon_0)$ 'or' $\sigma^e = E(Bq - \epsilon_0)$

where $\epsilon_0 = [\alpha\Delta t , \alpha\Delta t , 0]^T$, $\in = [\in_x , \in_y , \in_{xy}]^T$, and $\sigma = [\sigma_x , \sigma_y , \sigma_{xy}]^T$.

The elements thermal stresses along the *X*-axis, *Y*-axis and resultant direction are obtained using the results of displacements and temperatures at the nodes which are shown below in the Figures 6(a), 6(b) and 6(c), respectively.

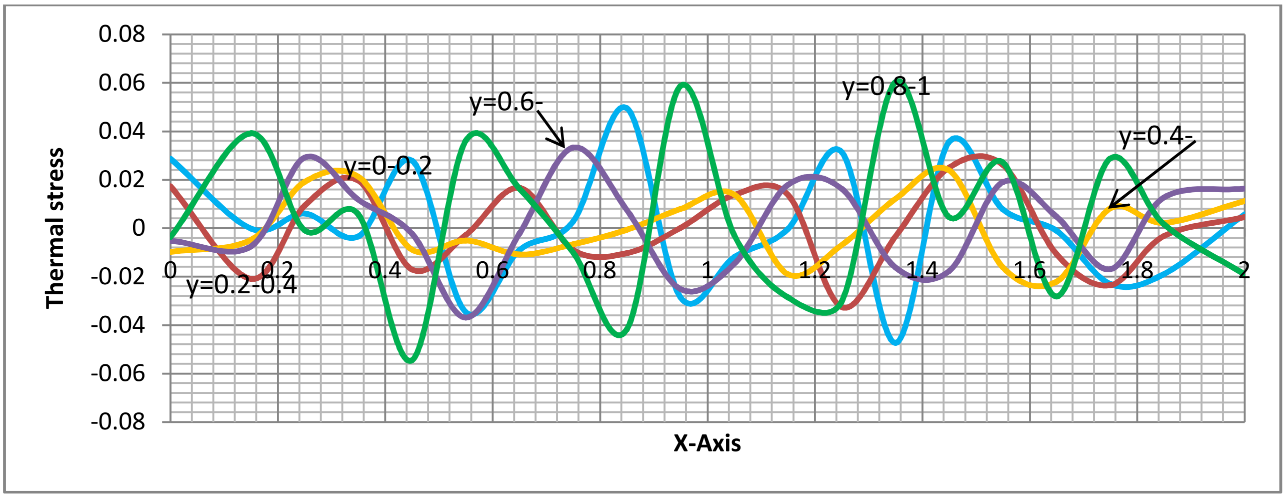

Figure 6a. Thermal Stress *(σ_x)* along *X* direction

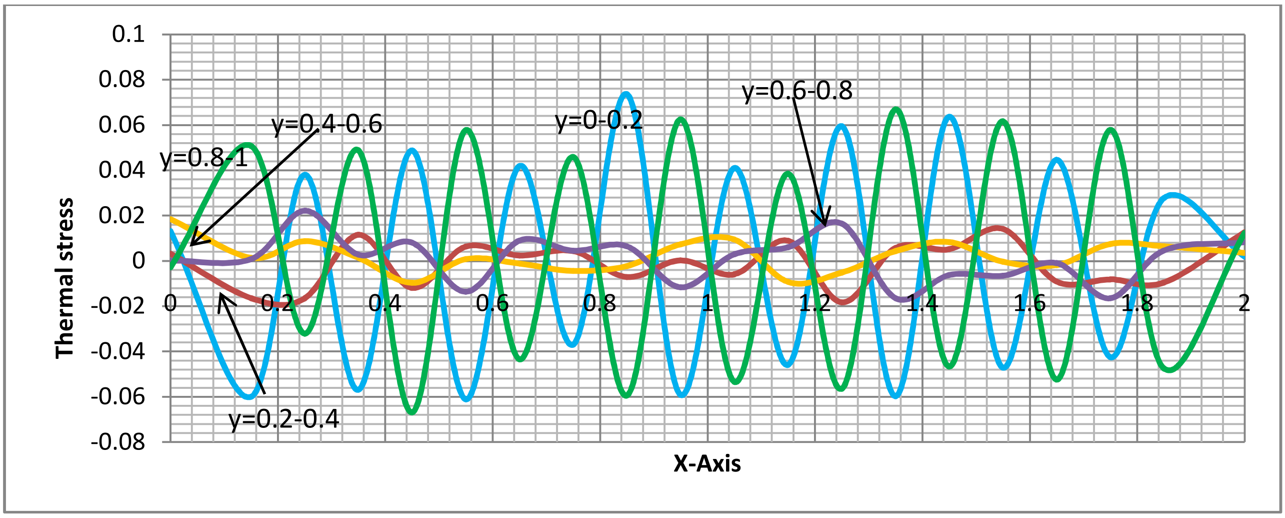

Figure 6b. Thermal Stress (σ_y) along *Y* direction

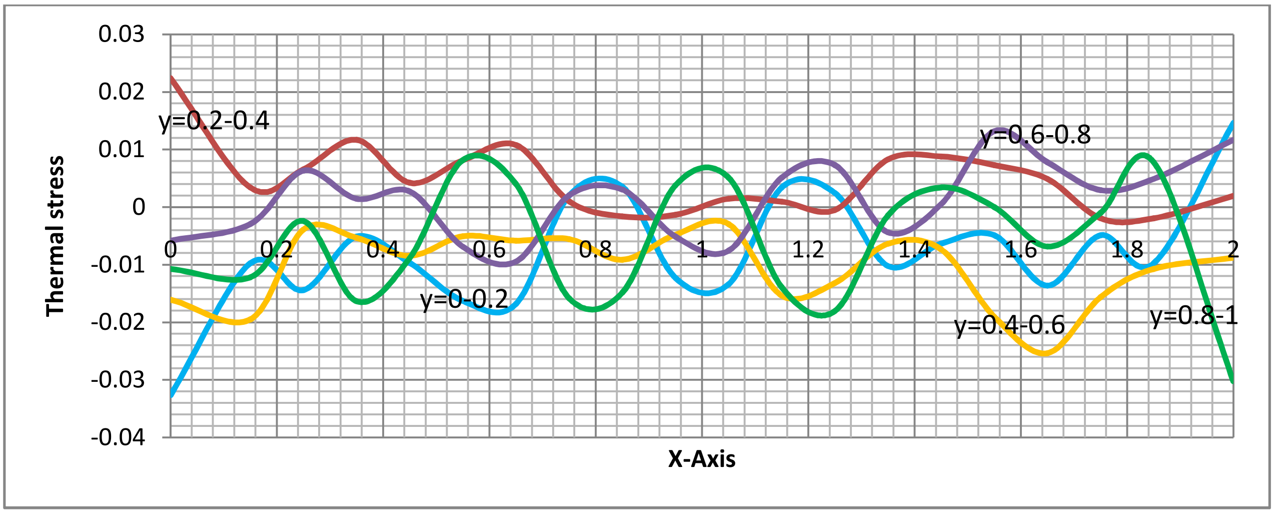

Figure 6c. Thermal Stress (σ_{xy}) along resultant direction

Conclusion

In this paper, an analytical FEM solution to transient heat conduction equation is presented. The attempt has been made for the heat transfer and thermal stress analysis of the isotropic rectangular plate with convective heat transfer in the form of partially distributed heat supply from the top surface and convection due to dissipation from bottom surface using finite element method. The Heaviside unit step function is used to describe partial distribution of heat supply. For the better accuracy of the numerical results, the large numbers of elements are taken for discretization. The Matlab programming is used for the determination of numerical values of temperatures, displacements and thermal stresses. In the Figures 3(a) and 3(b) the increase of temperature can be observed in the centre of the rectangular plate. The temperature varies from top to bottom but little high in the left part because of partial heat supply. From the Figure 5(a) it is observed that the displacement increases from top to bottom along X direction and very high at the upper left part, high at the right top part and low in the right lower part and Figure 5(b) shows the displacement along Y direction is very high in the top and bottom left corners, high in left part and low in right side of the plate. The development of thermal stresses seems extreme in the some portion of left half bottom, but high near the insulated surface and centre of the rectangular plate is shown in the Figures 6(a), and 6(b). In the Figure 6(c) thermal stress along resultant direction is seems high in the left portion and middle part but moderate near the top and bottom surfaces of the rectangular plate. The results presented in this paper should provide important information and useful guidance in many engineering problem.

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