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RESEARCH ARTICLE

COMPACT ORIENTED ANTI-INVARIANT SUBMANIFOLDS OF KAEHLERIAN MANIFOLD

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ABSTRACT

Anti-invariant (or totally real) submanifolds of Kaehlerian manifold have been studied by Blaer, Chen, Houh, Kon, Ludden Ogiue, Okumura, Yano and others. The purpose of this paper is to study a compact n-dimensional anti-invariant submanifold M immersed in n-dimensional complex projective space ($CP^n, n > 1$). First section contains some preliminaries and in section two we have pursued Kaehlerian manifold of dimension 2n and constant Holomorphic sectional curvature ($M^{2n}(c), n > 1$). Also some important theorems have been investigated. In third section we have discussed the compact oriented anti-invariant submanifold and its geodesic properties and obtained some results

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INTRODUCTION

Let M^{2n} be a Kaehlerian manifold of dimension greater than $2n$, where n is even and M an n-dimensional Riemannian manifold. Let J be the complex structure of M^{2n}. We define M a totally real submanifold of M^{2n} if M admits an isometric immersion into M^{2n} such that $J(T_x(M)) \subset T_x(M)^{\perp}$, where $T_x(M)$ and $T_x(M)^{\perp}$ denote the tangent space and the normal space of M at x respectively. Let h be the second fundamental form of M in M^{2n} and denote by S the square of length of the second fundamental form h. Now, Chen-Ogiue [9] have define and prove the following:

Definition 1: Let M be an n-dimensional compact anti-invariant minimal submanifold immersed in $M^{2n}(c)$, if $S < \frac{n(n+1)}{4(2n-1)}$ then M is totally geodesic.

Definition 2: Let M be an n-dimensional anti-invariant minimal submanifold immersed in $M^{2n}(c)$. If the sectional curvature of M is constant, then M is dither totally geodesic or has nonpositive sectional curvature. Moreover, if the second fundamental form of the immersion is parallel then M is totally geodesic or flat. Moreover, Ludden-Okumura-Yano [11] studied an n-dimensional anti-invariant minimal submanifold M of CP^n satisfying $S = \frac{n(n+1)}{2n-1}$, where CP^n denotes an n-dimensional complex projective space of constant holomorphic sectional curvature. Let a local field of orthogonal frames $l_1, \dots\dots, l_{2n}$ in M^{2n} which are tangent to M. Denote Jl_i by l_{i^*} and let $A^1, \dots\dots, A^{2n}$ be the field of dual frames with respect to the frame field of M^{2n}, then the structure equations of M^{2n} are given by

(1.1) $$dA^p = -A^p_q \wedge A^q,$$

(1.2) $$A^p_q + A^q_p = 0,\ A^u_v = A^{u^*}_{v^*},\ A^{u}_{v^*} = A^{v}_{u^*},$$

(1.3) $$dA^p_q = -A^p_r \wedge A^r_q + f^p_q, f^p_q = \frac{1}{2} K^p_{qrs} A^r \wedge A^s$$
$$K^p_{qrs} + K^p_{qsr} = 0$$

When we restrict these forms to M. we have

(1.4) $$A^i = 0.$$

Since $dA^i = -A^i_u \wedge A^u = 0$, by Cartan's lemma we can write

(1.5) $$A^i_u = h^i_{uv} A^u,\ h^i_{uv} = h^i_{vu}$$

and from (1.2) it follows that

(1.6) $$h^{u^*}_{vw} = h^{v^*}_{uw}$$

From these formulas we obtain the following structure equations of M:

(1.7) $$dA^u = -A^u_v \wedge A^v, , A^u_v + A^v_u = 0$$

(1.8)
$$dA^u_v = -A^u_w \wedge A^w_v + \Omega^u_v, \Omega^u_v = \frac{1}{2} R^u_{vwx} A^w \wedge A^x.$$

(1.9) $$R^u_{vwx} = K^u_{vwx} + \textstyle\sum_i (h^i_{uw} h^i_{vx} - h^i_{ux} h^i_{vw}).$$

(1.10) $$dA^i_j = -A^i_k \wedge A^k_j + \Omega^i_j, \Omega^i_j = \frac{1}{2} R^i_{jwx} A^w \wedge A^x,$$

(1.11) $$R^i_{jwx} = K^i_{jwx} + \textstyle\sum_i (h^i_{uw} h^j_{ux} - h^i_{ux} h^j_{uw}).$$

The forms (u^v_u) define the Riemannian connection of M, and the form (u^i_j) the connection induced in the normal bundle of M. From (1.2) and (1.5) it follows that

(1.12) $$h^{u^*}_{vw} = h^{v^*}_{uw} = h^{w^*}_{uv}$$

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where we have written $h^{u}_{\bar{v}w}$ in place of $h^{u}_{\dot{v}w}$ to simplify the notation. The second fundamental form of M is represented by $h^{i}_{uv}A^{u}A^{v}e_{i}$, and is sometimes denoted by its components h^{i}_{uv}. If the second fundamental form is of the form $\frac{\delta_{uv}(\sum_w h^{i}_{ww}e_i)}{n}$, then M is said to be totally umbilical. If h^{i}_{uv} is of the form $h^{i}_{uv} = \frac{(\sum_w h^{i}_{ww})\delta_{uv}}{n}$, then M is said to be umbilical with respect to e_i. We call $\frac{(\sum_w h^{i}_{ww}e_i)}{n}$ the mean curvature vector of M, and M is said to be minimal if its mean curvature vector vanishes identically, i.e., $\sum_w h^{i}_{ww} = 0$ for all i. We define the covariant derivative h^{i}_{uvw} of h^{i}_{uv} by

$$h^{i}_{uvw}A^{w} = dh^{i}_{uv} - h^{i}_{ux}A^{x}_{v} - h^{i}_{xv}A^{x}_{u} + h^{j}_{uv}A^{i}_{j} \tag{1.13}$$

The Laplacian Δh^{i}_{uv} of h^{i}_{uv} is defined to be

$$\Delta h^{i}_{uv} = \sum_w h^{i}_{uww}, \tag{1.14}$$

where we have defined h^{i}_{uvwx} by

$$h^{i}_{uvwx}A^{x} = dh^{i}_{uvw} - h^{i}_{xvw}A^{x}_{u} - h^{i}_{uxw}A^{x}_{v} - h^{i}_{uvx}A^{x}_{w} + h^{j}_{uvw}A^{i}_{j}. \tag{1.15}$$

In the sequel we assume that the second fundamental form of M satisfies equations of Codazzi:

$$h^{i}_{uvw} - h^{i}_{uwv} = 0. \tag{1.16}$$

Then, from (1.15), we have

$$h^{i}_{uvwx} - h^{i}_{uvxw} = h^{i}_{ux}R^{x}_{vwx} + h^{i}_{yv}R^{y}_{uwx} - h^{j}_{uv}R^{i}_{jwx}. \tag{1.17}$$

On the other hand, (1.14) and (1.15) imply that

$$\Delta h^{i}_{uv} = \sum_w h^{i}_{uvww} = \sum_w h^{i}_{wuvw}. \tag{1.18}$$

If M^{2n} is locally symmetric, then we have the following equation (Braidi-Hsiung)

$$\begin{aligned} \sum_{i,u,v} h^{i}_{uv} = \sum_{i,u,v,w}\big(&h^{i}_{uv}h^{i}_{wwuv} - K^{i}_{uvj}h^{i}_{uv}h^{j}_{ww} + 4K^{i}_{jwu}h^{j}_{vw}h^{i}_{uv} - K^{i}_{wjw}h^{i}_{uv}h^{j}_{uv} + \\ &+2K^{y}_{wuw}h^{i}_{yv}h^{i}_{uv} + 2K^{y}_{uvw}h^{i}_{yw}h^{i}_{uv}\big) - \sum_{i,j,u,v,w,x}\big[\big(h^{i}_{uw}h^{j}_{vw} - \\ &-h^{i}_{vw}h^{j}_{uw}\big)\big(h^{i}_{ux}h^{j}_{vx} - h^{i}_{vx}h^{j}_{ux}\big) + h^{i}_{uv}h^{i}_{wx}h^{j}_{uv}h^{j}_{wx} - h^{i}_{uv}h^{i}_{wu}h^{j}_{wv}h^{j}_{xx}\big] \end{aligned} \tag{1.19}$$

Compact Oriented Anti-invariant Submanifolds

Let us suppose $M^{2n}(c)$ is a Kaehlerian manifold of dimension 2n of constant holomorphic sectional curvature 'c'. Then the curvature tensor of M^{2n} is given by

(2.1)

$$K^{p}_{qrs} = \frac{1}{4}c(\delta_{pr}\delta_{qs} - \delta_{ps}\delta_{qr} + J_{pr}J_{qs} - J_{ps}J_{qr} + 2J_{pq}J_{rs}),$$

where δ_{pr} denote the Kronecker deltas. Let M be an n-dimensional anti-invariant submanifold immersed in $[M^{2n}(c)]^{n}$. From the condition on the dimension of M and M^{2n} it follows that $l_1, \ldots\ldots, l_x$ is a frame for $T_x(M)^{\perp}$. In view of this and using (1.2), (1.5) and (2.1) we reduce (1.18) to

$$\begin{aligned} \sum_{i,u,v} h^{i}_{uv}\Delta h^{i}_{uv} = &\sum_{i,u,v,w} h^{i}_{uv}h^{i}_{wwuv} + \tfrac{1}{4}(n+1)c\sum_{i,u,v} h^{i}_{uv}h^{i}_{uv} - \tfrac{1}{2}c\sum_i\big(\sum_i h^{i}_{uu}\big)^2 + \\ &+\sum_{i,j,u,v,w,x}\big(h^{i}_{uv}h^{j}_{vw}h^{i}_{wu}h^{j}_{xx} - h^{i}_{uv}h^{j}_{uv}h^{i}_{wx}h^{j}_{wx}\big) - \\ &\sum_{i,j,u,v,w,x}\big(h^{i}_{uw}h^{j}_{wv} - h^{j}_{uw}h^{i}_{wv}\big)\big(h^{i}_{ux}h^{j}_{xv} - h^{j}_{ux}h^{i}_{xv}\big). \end{aligned} \tag{2.2}$$

For each a, let H_a denote the symmetric matrix (h^{i}_{uv}).then (2.2) can be written as

(2.3)

$$\sum_{i,u,v} h^{i}_{uv}\Delta h^{i}_{uv} = \sum_{i,u,v,w} h^{i}_{uv}h^{i}_{wwuv} + \sum_i\left[\tfrac{1}{4}(n+1)c\,trH_i^2 - \tfrac{1}{2}c(trH_i)^2\right] + \sum_{i,j}\{tr(H_iH_j - H_jH_i)^2 - [tr(H_iH_j)]^2 + trH_j tr(H_iH_jH_i)\}$$

where trH_i^2 denote the trace of the square matrix H_i^2 i.e. sum of the element of main diagonal of a square matrix. Equation (2.3) was obtained by Chen-Ogiue [9] for an anti-invariant minimal submanifold M^n immersed in $[M^{2n}(c)]^n$. Now set

$$S_{ij} = \sum_{uv} h^{i}_{uv}h^{j}_{uv},\ S_i = S_{ii},\ S = \sum_i S_i,$$

so that S_{ij} is symmetric $(n \times n)$-matrix and can be assumed to be diagonal for a suitable choice of $l_{n+1}, \ldots\ldots, l_{2n}$, and S is the square of the length of the second fundamental form h^{i}_{uv} of M. Since $trD^2 = \sum_{u,v}(a_{uv})^2$ is independent of the choice of a frame, for any symmetric $D = (a_{uv})$ we can rewrite (2.3) as

(2.4)

$$\begin{aligned} \sum_{i,u,v} h^{i}_{uv}\Delta h^{i}_{uv} = &\sum_{i,u,v,w} h^{i}_{uv}h^{i}_{wwuv} + \tfrac{1}{4}(n+1)cS - \sum_i S_i^2 + \sum_{i,j} tr\,(H_iH_j - H_jH_i)^2 - \\ &-\tfrac{1}{2}c\sum_i(trH_i)^2 + \sum_{i,j} trH_j tr(H_iH_jH_i). \end{aligned}$$

Now we have the following lemma

Lemma (I): Let D and E be symmetric $(n \times n)$ −metrics. Then

$$-tr(DE - ED)^2 \le 2trD^2trE^2$$

and the equality holds for non-zero matrices D and E if and only if D and E can be transformed simultaneously by an orthogonal matrix simultaneously into scalar multiples of $\bar{D}$ and $\bar{E}$ respectively, where

$$\bar{D} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \bar{E} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Moreover, if D_1, D_2, D_3 are symmetric (n, n) −matrices such that

$$-tr(D_iD_j - D_jD_i)^2 = 2tr\,D_i^2trD_j^2,\ 1 \le i, j \le 3,\ i \ne j,$$

then at least one of the matrices D_i must be zero.
by using Lemma 1, we have the following inequality which plays an important role in the sequel.

$$\begin{aligned} -\sum_{i,j} tr\big(H_iH_j - H_jH_i\big)^2 + \sum_i S_i^2 - \tfrac{1}{4}(n+1)cS \le 2\sum_{i\ne j} S_iS_j + \sum_i S_i^2 - \\ -\frac{1}{4}(n+1)cS = \left[\left(2 - \frac{1}{n}\right)S - \frac{1}{4}(n+1)c\right]S - \frac{1}{n}\sum_{i>j}(S_i - S_j)^2, \end{aligned} \tag{2.5}$$

Then (2.4) and (2.5) imply that

$$-\sum_{i,u,v} h^{i}_{uv}\Delta h^{i}_{uv} \le B - \sum_{i,u,v,w} h^{i}_{uv}h^{i}_{wwuv}\ , \tag{2.6}$$

where

(2.7) $B = \left[\left(2-\frac{1}{n}\right)S - \frac{1}{4}(n+1)c\right]S + \frac{1}{2}c\sum_i (trH_i)^2 - \sum_{i,j} tr\, H_j tr(H_iH_jH_i)$.

Theorem (2.1): Let M be an n-dimensional compact oriented anti-invariant submanifold immersed in Kaehlerian manifold of dimension 2n and constant Holomorphic sectional curvature $[M^{2n}(c)]^n$. Then

(2.8) $$\int_M [B - \sum_i (trH_i)\Delta(trH_i)]dV \geq 0,$$

where dV denotes the volume element of M.

Proof: First we obtain

$$\int_M \sum_{i,u,v,w} (h^i_{uvw})^2\, dV - -\int_M \sum_{i,u,v} h^i_{uv}\Delta h^i_{uv}\, dV \geq 0.$$

On the other hand, we have [8]

$$\int_M \sum_{i,u,v,w} h^i_{uv} h^i_{wwuv}\, dV = \int_M \sum_i (trH_i)\Delta(trH_i)dV.$$

From these equations and (2.6) follows the inequality

(2.9) $$\int_M [B - \sum_i (trH_i)\Delta(trH_i)]dV \geq \int_M \sum_{i,u,v,w} (h^i_{uvw})^2\, dV \geq 0,$$

which is same as equation (2.8).

Theorem (2.2): Let M be an n-dimensional compact oriented anti-invariant minimal submanifold immersed in Kaehlerian manifold of dimension $2n$ and constant Holomorphic sectional curvature i.e. $M^{2n}(c)$. Then

(2.10) $$\int_M \left[\left(2-\frac{1}{n}\right)S - \frac{1}{4}(n+1)c\right]S\, dV \geq 0.$$

This is the special case of theorem (2.1) which was proved essentially by Chen-Ogiue [9].

2. Compact oriented anti-invariant submanifold in a totally geodesic

Now we assume that M is an n-dimensional compact oriented totally real submanifold immersed in $[M^{2n}(c)]^n, n > 1$ and that M is not totally geodesic in M^{2n} but satisfies

(3.1) $$\int_M [B - \sum_i (trH_i)\Delta(trH_i)]dV = 0.$$

Then (2.9) implies that $h^i_{uvw} = 0$, i.e., the second fundamental form of M is covariant constant, so that $\Delta h^i_{uv} = 0$, and all terms on both sides of (2.6) vanish. It follows that inequalities (2.4) and (2.5) imply

(3.2) $$\frac{1}{n}\sum_{i>j}(S_i - S_j)^2 = 0,$$

(3.3) $$-tr(H_iH_j - H_jH_i)^2 = 2tr\, H_i^2 tr H_j^2$$

For any $i \neq j$. Then by Lemma 1 we may assume that $H_i = 0$ for $i = n+3, \dots, 2n$, which shows that $S_i = 0$. But by (3.2) we case that $S_i = S_j$ for any i, j. Since M is not totally geodesic, $n = 2$ and therefore by using Lemma 1 we can assume that

(3.4) $$H_{n+1} = \lambda \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, H_{n-2} = \mu \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

From this it follows that M is a minimal surface immersed in $[M^{2n}(c)]^n$. Since the second fundamental form h of M^2 is covariant constant, the sectional curvature of M^2 is constant and hence M^2 is flat by definition 2. On the other hand, by using (1.13) we obtain

(3.5) $$dh^i_{uv} = h^i_{ux}A^x_v + h^i_{xv}A^x_u - h^j_{uv}A^i_j.$$

Setting $i - 3, u - 1, v - 2,$ we see that $d\lambda - dh^3_{12} - 0$, which means that λ is constant. Similarly setting $i = 4$ and $u = v = 1$, we see that μ is constant. By (3.2) we get $\lambda^2 = \mu^2$, and since $S = \frac{1}{2}c$ we have $\lambda^2 + \mu^2 = \frac{1}{4}c$ so that $\lambda^2 = \frac{1}{8}c$. Since M is not totally geodesic, we may assume that $c > 0$ and $-\lambda = \mu = \frac{1}{2}\sqrt{\frac{c}{2}}$. Then (1.5) and (3.4) imply

$$A^3_1 = \lambda A^2, A^3_2 = \lambda A^1, A^4_1 = \mu A^1, A^4_2 = -\mu A^2.$$

On the other hand, setting $i = 3, u = v = 1$ in (3.5), we have $A^3_4 = \frac{2\lambda}{\mu}A^2_1 = 2A^1_2$. Hence we obtain the following

Theorem (3.1): Let M be an $n-$dimensional compact oriented anti-invariant submanifold immersed in $[M^{2n}(c)]^n, n > 1$ s.t. M is not totally geodesic but condition (3.1) exist. Then M is a flat surface minimally immersed in $[M^{2n}(c)]^2$ and w. r. t. an adapted dual orthonormal frame field A^1, A^2, A^3, A^4, the connection form (A^p_q) of $[M^{2n}(c)]^2$, restricted to M, is given by

$$\begin{bmatrix} 0 & A^1_2 & -\lambda A^2 & -\mu A^1 \\ -A^1_2 & 0 & -\lambda A^1 & \mu A^2 \\ \lambda A^2 & \lambda A^1 & 0 & 2A^1_2 \\ \mu A^1 & -\mu A^2 & -2A^1_2 & 0 \end{bmatrix}, -\lambda = \mu = \frac{1}{2}\sqrt{\frac{c}{2}}.$$

Here, we take an n-dimensional complex projective space CP^n of constant holomorphic sectional curvature 4 as an ambient space. Then the above theorem implies.

Theorem (3.2): Let M be an n-dimensional compact oriented totally real submanifold immersed in CP^n, $n > 1$, such that M is not totally geodesic but satisfies (3.1). Then $n - 2$ and $M = S^1 \times S^1$.

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