Available online at http://www.journalcra.com

International Journal of Current Research
Vol. 18, Issue, 03, pp.36432-36436, March, 2026
DOI: https://doi.org/10.24941/ijcr.50292.03.2026

INTERNATIONAL JOURNAL OF CURRENT RESEARCH

ISSN: 0975-833X

RESEARCH ARTICLE

ON A NEW DYNAMICAL SYSTEMS ON GENERALIZED–QUATERNIONIC KÄHLER MANIFOLDS BY USING A CANONICAL LOCAL BASIS

Gebreel Mohammed Khur Baba Gebreel , Ibrahim Yousif Ibrahim Abad Alrhman and Eltayeb Awad Adam

Department of Mathematics, Faculty of Education, West Kordufan University, Khartoum-Sudan

ARTICLE INFO

Article History:
Received 15th December, 2025
Received in revised form
25th January, 2026
Accepted 18th February, 2026
Published online 29th March, 2026

Keywords:
Preliminaries; Generalized-Quaternionic $k\ddot{a}hler$ geometry; Lagrangian mechanical systems.

Corresponding author:
Gebreel Mohammed Khur Baba Gebreel

ABSTRACT

In this paper, we study a new formula of Lagrangian dynamical system on generalized-quaternionic $k\ddot{a}hler$ Manifolds by using vector field: $Z = \sum_{a=0}^{3} Z^{an+} \frac{\partial}{\partial x_{an+i}}$ in a canonical local basis $\{H\}$ of V. Finally, the geometrical – Physical results related to generalized-quaternionic $k\ddot{a}hler$ mechanical systems are also given.

Citation: Gebreel Mohammed Khur Baba Gebreel , Ibrahim Yousif Ibrahim Abad Alrhman and Eltayeb Awad Adam. 2026. "On A new Dynamical Systems On Generalized–Quaternionic Ka¨hler Manifolds By Using A canonical Local Basis". ***International Journal of Current Research, 18, (03), 36432-36436.***

INTRODUCTION

Modern differential geometry plays an important role to explain the dynamics of Lagrangians. So , if Q is an $m - dimensional$ configuration manifold and $L : TQ \rightarrow R$ is a regular Lagrangian function , then it is well-known that there is a unique vector field ξ on TQ such that the dynamic equation is given by

$$i_{\xi}\Phi_L = dE_L \qquad \rightarrow \qquad (1)$$

Where $\Phi_L = -dd_J L$ indicates the symplectic form , $J : TQ \rightarrow TQ$ respectively an almost tangent or complex or paracomplex structure such that $JoJ = 0$ or $JoJ = -1$ or $JoJ = 1$. Also $E_L = V(L) - L$ is an energy function and $V = J(\xi)$ a Liouville vector field. The triple (TQ , Φ_L , ξ) is called Lagrangian system on the tangent bundle . Also , modern differential geometry provides a good framework in which develops the dynamics of Hamiltonians. Therefore, if Q is an m-dimensional configuration manifold and $H : T^*Q \rightarrow R$ is a regular Hamiltonian function, then there is a unique vector field X on T^*Q such that the dynamic equation is given by :

$$i_X\Phi = dH \qquad \rightarrow \qquad (2)$$

Where Φ indicates the canonical symplectic form. The triple (T^*Q, Φ, X) is called Hamiltonian system on the cotangent bundle T^*Q. Nowadays , there are many studies about Lagrangian and Hamiltonian dynamics, mechanics, formalisms, Systems and equations (1, 2) and therein. There are real, complex, paracomplex and other analogues. As known it is possible to produce different analogous in different spaces. Quaternions were invented by Sir William Rowan Hamiltonian as an extension to the complex numbers. Hamiltonian's defining relation is most succinctly written as:

$$i^2 = j^2 = k^2 = -1 , \qquad ijk = -1 .$$

Split quaternions are given by

$$i^2 = -1 \;,\; j^2 = 1 = k^2 \;, ijk = 1$$

Generalized-quaternions are defined as

$$i^2 = -a \;,\; j^2 = -b \;,\; k^2 = -ab \;,\; ijk = -ab$$

If it is compared to the calculus of vectors, quaternions have slipped into the realm of obscurity. They do however still find use in the computation of rotations. A lot of physical laws in classical, relativistic, and quantum mechanics can be written pleasantly by means of quaternions. Some physicists hope they will find deeper understanding of the universe by restating basis principles in terms of quaternion algebra (3, 4).

Preliminaries: In this paper, we present equations related to Lagrangian mechanical systems on a generalized-quaternionic $k\ddot{a}hler$ manifold, all mathematical objects and mappings are assumed to be smooth, i.e. infinitely differentiable and Einstein convention of summarizing is adopted. $\mathcal{F}(M), \varkappa(M) and\ \Lambda^1(M)$ denote the set of functions on M, the set of vector fields on M and the set of 1-forms on M, respectively.

Theorem

Let f be differentiable ϕ , ψ are 1-forme, then (5 , 6):

- $d(f\phi) = df\Lambda\phi + fd\phi$
- $d(\phi\Lambda\psi) = d\phi\Lambda\psi - \phi\Lambda d\psi$

Definition (Kronecker's delta)

A kronecker's delta denote by δ and defined as follows (5):

$$\delta_i^j = \begin{cases} 1 \; ; & if \quad i = j \\ 0 \; ; & if \quad i \neq j \end{cases}$$

Generalized-Quaternionic $k\ddot{a}hler$ Manifolds: A generalized almost quaternion structure on the manifold M is a sub bundle of the bundle of endomorphisms of the tangent bundle of M, whose standard fiber is the algebra of generalized-quaternions. A generalized almost quaternion structure on a pseudo-Riemannian manifold is called a generalized-quaternion Hermitian if the following conditions hold:

(i) The endomorphisms $F \,, G \; and \; H \; of \; T_x M$ satisfy

$$F^2 = -aI,\; G^2 = -bI \,,\; H^2 = -abI, FG = H \,,\; GH = bF \,, HF = aG \quad \rightarrow \tag{3}$$

(ii) The compatibility equations are given by, for $X \,, Y \in T_x M$

$$g(FX,FY) = ag(X,Y),\; g(GX,GY) = bg(X,Y),\; g(HX,HY) = abg(X,Y) \;\rightarrow \tag{4}$$

$$\begin{aligned} &F\left(\frac{\partial}{\partial x_i}\right) = a\frac{\partial}{\partial x_{n+i}} \;, && G\left(\frac{\partial}{\partial x_i}\right) = -b\frac{\partial}{\partial x_{2n+i}} \;, && H\left(\frac{\partial}{\partial x_i}\right) = -ab\frac{\partial}{\partial x_{3n+i}} \\ &F\left(\frac{\partial}{\partial x_{n+i}}\right) = -a\frac{\partial}{\partial x_i} \;, && G\left(\frac{\partial}{\partial x_{n+i}}\right) = b\frac{\partial}{\partial x_{3n+i}} \;, && H\left(\frac{\partial}{\partial x_{n+i}}\right) = -ab\frac{\partial}{\partial x_{2n+i}} \\ &F\left(\frac{\partial}{\partial x_{2n+i}}\right) = a\frac{\partial}{\partial x_{3n+i}} \;, && G\left(\frac{\partial}{\partial x_{2n+i}}\right) = -b\frac{\partial}{\partial x_i} \;, && H\left(\frac{\partial}{\partial x_{2n+i}}\right) = -ab\frac{\partial}{\partial x_{n+i}} \\ &F\left(\frac{\partial}{\partial x_{3n+i}}\right) = -a\frac{\partial}{\partial x_{2n+i}} \;, && G\left(\frac{\partial}{\partial x_{3n+i}}\right) = b\frac{\partial}{\partial x_{n+i}} \;, && H\left(\frac{\partial}{\partial x_{3n+i}}\right) = -ab\frac{\partial}{\partial x_i} \end{aligned} \tag{5}$$

A canonical local basis $\{F^* \,, G^* \,, H^*\} \; of \; V^*$ of the cotangent space $T^*(M)$ of a manifold M satisfies the condition as follows:

$$F^{*^2} = -aI \,, G^{*^2} = -bI \,, H^{*^2} = -abI \,, F^*G^* = H^* \,,\; G^*H^* = bF \;,\; H^*F^* = aG^* \;\rightarrow \tag{6}$$

Defining by

$$\begin{aligned} &F^*(dx_i) = adx_{n+i} \;, && G^*(dx_i) = -bdx_{2n+i} \;, && H^*(dx_i) = -abdx_{3n+i} \\ &F^*(dx_{n+i}) = -adx_i \;, && G^*(dx_{n+i}) = bdx_{3n+i} \;, && H^*(dx_{n+i}) = -abdx_{2n+i} \;\rightarrow \\ &F^*(dx_{2n+i}) = adx_{3n+i} \;, && G^*(dx_{2n+i}) = -bdx_i \;, && H^*(dx_{2n+i}) = -abdx_{n+i} \end{aligned} \tag{7}$$

$F^*(dx_{3n+i}) = -adx_{2n+i}\ ,\ G^*(dx_{3n+i}) = bdx_{n+i}\ ,\ H^*(dx_{3n+i}) = -abdx_i$

Lagrangian Mechanical Systems: Here, we obtain Euler-Lagrange equation for quantum and classical mechanics by means of a canonical local basis $\{F, G, H\}$ $of\ V$ on a generalized-quaternionic $k\ddot{a}hler$ manifold (M, g, V).

First:

$$a\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_i}\right) + \frac{\partial L}{\partial x_{n+i}} = 0\ ,\ a\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{n+i}}\right) - \frac{\partial L}{\partial x_i} = 0\ ,$$
$$a\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{2n+i}}\right) + \frac{\partial L}{\partial x_{3n+i}} = 0\ ,\ a\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{3n+i}}\right) - \frac{\partial L}{\partial x_{2n+i}} = 0 \quad \rightarrow \quad (8)$$

Second:

$$b\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_i}\right) + \frac{\partial L}{\partial x_{2n+i}} = 0\ ,\ b\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{n+i}}\right) - \frac{\partial L}{\partial x_{3n+i}} = 0$$
$$b\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{2n+i}}\right) + \frac{\partial L}{\partial x_i} = 0\ ,\ b\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{3n+i}}\right) - \frac{\partial L}{\partial x_{n+i}} = 0 \quad 9,10) \rightarrow \quad (9)$$

Third, let H take a local basis element on the generalized-quaternionic $k\ddot{a}hler$ manifold (M, g, V), and $\{x_i, x_{n+i}, x_{2n+i}, x_{3n+i}\}$ be its coordinate functions. Let semispray be the vector field Y determined by

$$Z = Z^i\frac{\partial}{\partial x_i} + Z^{n+i}\frac{\partial}{\partial x_{n+i}} + Z^{2n+i}\frac{\partial}{\partial x_{2n+i}} + Z^{3n+i}\frac{\partial}{\partial x_{3n+i}} \quad (9). \rightarrow \quad (10)$$

Where $Z^i = \dot{x}_i\ ,\ Z^{n+i} = \dot{x}_{n+i}\ ,\ Z^{2n+i} = \dot{x}_{2n+i}\ ,\ Z^{3n+i} = \dot{x}_{3n+i}$.

This equation (10) can be written concise manner:

$$Z = \sum_{a=0}^{3} Z^{an+i}\frac{\partial}{\partial x_{an+i}} \quad \rightarrow \quad (11)$$

And the dot indicates the derivative with respect to time t. The vector field defined by:

$$V_H = H(Z) = -abZ^i\frac{\partial}{\partial x_{3n+i}} - abZ^{n+i}\frac{\partial}{\partial x_{2n+i}} - abZ^{2n+i}\frac{\partial}{\partial x_{n+i}} - abZ^{3n+i}\frac{\partial}{\partial x_i}\ :\ \mathcal{F}(M) \rightarrow \Lambda^1\mathrm{M} \ \rightarrow \quad (12)$$

Is named a Liouville vector field on the generalized-quaternionic $k\ddot{a}hler$manifold (M, g, V), for H, the closed generalized-quaternionic $k\ddot{a}hler$ form is the closed 2-form given by $\Phi_L^H = -dd_HL$ such that

$$d = \frac{\partial}{\partial x_i}dx_i + \frac{\partial}{\partial x_{n+i}}x_{n+i} + \frac{\partial}{\partial x_{2n+i}}x_{2n+i} + \frac{\partial}{\partial x_{3n+i}}x_{3n+i} \quad \rightarrow \quad (13)$$

This equation (13) can be written concise manner

$$d = \sum_{a=0}^{3}\frac{\partial}{\partial x_{an+i}}dx_{an+i} \quad \rightarrow \quad (14)$$

$$d_HL = -ab\frac{\partial L}{\partial x_{3n+i}}dx_i - ab\frac{\partial L}{\partial x_{2n+i}}dx_{n+i} - ab\frac{\partial L}{\partial x_{n+i}}dx_{2n+i} - ab\frac{\partial L}{\partial x_i}dx_{3n+i}\ :\ \mathcal{F}(M) \rightarrow \Lambda^1\mathrm{M} \ \rightarrow \quad (15)$$

Then we have:

$$\begin{aligned}\Phi_L^H = {} & ab\frac{\partial^2 L}{\partial x_j\partial x_{3n+i}}dx_j \wedge dx_i + ab\frac{\partial^2 L}{\partial x_j\partial x_{2n+i}}dx_j \wedge dx_{n+i} + ab\frac{\partial^2 L}{\partial x_j\partial x_{n+i}}dx_j \wedge dx_{2n+i} \\ & +ab\frac{\partial^2 L}{\partial x_j\partial x_i}dx_j \wedge dx_{3n+i} + ab\frac{\partial^2 L}{\partial x_{n+j}\partial x_{3n+i}}dx_{n+j} \wedge dx_i + ab\frac{\partial^2 L}{\partial x_{n+j}\partial x_{2n+i}}dx_{n+j} \wedge dx_{n+i} \\ & +ab\frac{\partial^2 L}{\partial x_{n+j}\partial x_{n+i}}dx_{n+j} \wedge dx_{2n+i} + ab\frac{\partial^2 L}{\partial x_{n+j}\partial x_i}dx_{n+j} \wedge dx_{3n+i} + ab\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{3n+i}}dx_{2n+j} \wedge dx_i \\ & ab\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{2n+i}}dx_{2n+j} \wedge dx_{n+i} + ab\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{n+i}}dx_{2n+j} \wedge dx_{2n+i} + ab\frac{\partial^2 L}{\partial x_{2n+j}\partial x_i}dx_{2n+j} \wedge dx_{3n+i} \\ & +b\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{3n+i}}dx_{3n+j} \wedge dx_i + ab\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{2n+i}}dx_{3n+j} \wedge dx_{n+i} + ab\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{n+i}}dx_{3n+j} \wedge dx_{2n+i} \\ & +ab\frac{\partial^2 L}{\partial x_{3n+j}\partial x_i}dx_{3n+j} \wedge dx_{3n+i}\end{aligned}$$

$$\because\ i_Z = Z^i\frac{\partial}{\partial x_i} + Z^{n+i}\frac{\partial}{\partial x_{n+i}} + Z^{2n+i}\frac{\partial}{\partial x_{2n+i}} + Z^{3n+i}\frac{\partial}{\partial x_{3n+i}}$$

Then we calculate

$$
\begin{aligned}
i_Z\Phi_L^H &= abZ^i\frac{\partial^2 L}{\partial x_j\partial x_{3n+i}}\delta_i^j dx_i - abZ^i\frac{\partial^2 L}{\partial x_j\partial x_{3n+i}}dx_j + abZ^i\frac{\partial^2 L}{\partial x_j\partial x_{2n+i}}\delta_i^j dx_{n+i} - abZ^{n+i}\frac{\partial^2 L}{\partial x_j\partial x_{2n+i}}dx_j \\
&+a\,bZ^i\frac{\partial^2 L}{\partial x_j\partial x_{n+i}}\delta_i^j dx_{2n+i} - abZ^{2n+i}\frac{\partial^2 L}{\partial x_j\partial x_{n+i}}dx_j + abZ^i\frac{\partial^2 L}{\partial x_j\partial x_i}\delta_i^j dx_{3n+i} - abZ^{3n+i}\frac{\partial^2 L}{\partial x_j\partial x_i}dx_j \\
&+abZ^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{3n+i}}\delta_{n+i}^{n+j}dx_i - abZ^i\frac{\partial^2 L}{\partial x_{n+j}\partial x_{3n+i}}dx_{n+j} + abZ^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{2n+i}}\delta_{n+i}^{n+j}dx_{n+i} \\
&-abZ^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{2n+i}}dx_{n+j} + abZ^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{n+i}}\delta_{n+i}^{n+j}dx_{2n+i} - abZ^{2n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{n+i}}dx_{n+j} \\
&+abZ^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_i}\delta_{n+i}^{n+j}dx_{3n+i} - abZ^{3n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_i}dx_{n+j} + abZ^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{3n+i}}\delta_{2n+i}^{2n+j}dx_i \\
&-abZ^i\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{3n+i}}dx_{2n+j} + abZ^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{2n+i}}\delta_{2n+i}^{2n+j}dx_{n+i} - abZ^{n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{2n+i}}dx_{2n+j} \\
&+abZ^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{n+i}}\delta_{2n+i}^{2n+j}dx_{2n+i} - abZ^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{n+i}}dx_{2n+j} + abZ^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_i}\delta_{2n+i}^{2n+j}dx_{3n+i} \\
&-abZ^{3n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_i}dx_{2n+j} + abZ^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{3n+i}}\delta_{3n+i}^{3n+j}dx_i - abZ^i\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{3n+i}}dx_{3n+j} \\
&+abZ^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{2n+i}}\delta_{3n+i}^{3n+j}dx_{n+i} - abZ^{n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{2n+i}}dx_{3n+j} + abZ^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{n+i}}\delta_{3n+i}^{3n+j}dx_{2n+i} \\
&-abZ^{2n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{n+i}}dx_{3n+j} + abZ^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_i}\delta_{3n+i}^{3n+j}dx_{3n+i} - abZ^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_i}dx_{3n+j}
\end{aligned}
$$

The Energy function is

$$E_L^H = V_H(L) - L$$

$$E_L^H = -abZ^i\frac{\partial L}{\partial x_{3n+i}} - abZ^{n+i}\frac{\partial L}{\partial x_{2n+i}} - abZ^{2n+i}\frac{\partial L}{\partial x_{n+i}} - abZ^{3n+i}\frac{\partial L}{\partial x_i} - L \quad \rightarrow \tag{16}$$

$$
\begin{aligned}
dE_L^H &= -abZ^i\frac{\partial^2 L}{\partial x_j\partial x_{3n+i}}dx_j - abZ^{n+i}\frac{\partial^2 L}{\partial x_j\partial x_{2n+i}}dx_j - abZ^{2n+i}\frac{\partial^2 L}{\partial x_j\partial x_{n+i}}dx_j - abZ^{3n+i}\frac{\partial^2 L}{\partial x_j\partial x_i}dx_j - abZ^i\frac{\partial^2 L}{\partial x_{n+j}\partial x_{3n+i}}dx_{n+j} - \\
&abZ^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{2n+i}}dx_{n+j} - abZ^{2n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{n+i}}dx_{n+j} - abZ^{3n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_i}dx_{n+j} \\
&-abZ^i\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{3n+i}}dx_{2n+j} - abZ^{n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{2n+i}}dx_{2n+j} - abZ^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{n+i}}dx_{2n+j} \\
&-abZ^{3n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_i}dx_{2n+j} - abZ^i\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{3n+i}}dx_{3n+j} - abZ^{n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{2n+i}}dx_{3n+j} \\
&-abZ^{2n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{n+i}}dx_{3n+j} - abZ^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_i}dx_{3n+j} - \frac{\partial L}{\partial x_j}dx_j - \frac{\partial L}{\partial x_{n+j}}dx_{n+j} \\
&-\frac{\partial L}{\partial x_{2n+j}}dx_{2n+j} - \frac{\partial L}{\partial x_{3n+j}}dx_{3n+j}
\end{aligned}
$$

Using equation (1) and also considering an Integral curve of Z, we obtain the equation given by:

$$i_Z\Phi_L^H = dE_L^H$$

From which we get

$$
\begin{aligned}
&ab(Z^i\frac{\partial^2 L}{\partial x_j\partial x_{3n+i}}\delta_i^j + Z^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{3n+i}}\delta_{n+i}^{n+j} + Z^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{3n+i}}\delta_{2n+i}^{2n+j} + Z^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{3n+i}}\delta_{3n+i}^{3n+j})dx_i \\
&+ab(Z^i\frac{\partial^2 L}{\partial x_j\partial x_{2n+i}}\delta_i^j + Z^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{2n+i}}\delta_{n+i}^{n+j} + Z^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{2n+i}}\delta_{2n+i}^{2n+j} + Z^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{2n+i}}\delta_{3n+i}^{3n+j})dx_{n+i} \\
&ab(Z^i\frac{\partial^2 L}{\partial x_j\partial x_{n+i}}\delta_i^j + Z^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{n+i}}\delta_{n+i}^{n+j} + Z^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{n+i}}\delta_{2n+i}^{2n+j} + Z^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{n+i}}\delta_{3n+i}^{3n+j})dx_{2n+i} \\
&+ab(Z^i\frac{\partial^2 L}{\partial x_j\partial x_i}\delta_i^j + Z^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_i}\delta_{n+i}^{n+j} + Z^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_i}\delta_{2n+i}^{2n+j} + Z^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_i}\delta_{3n+i}^{3n+j})dx_{3n+i} \\
&= -(\frac{\partial L}{\partial x_j}dx_j + \frac{\partial L}{\partial x_{n+j}}dx_{n+j} + \frac{\partial L}{\partial x_{2n+j}}dx_{2n+j} + \frac{\partial L}{\partial x_{3n+j}}dx_{3n+j})
\end{aligned}
$$

If $i = j \implies \delta_i^j = 1, \delta_{n+i}^{n+j} = 1, \delta_{2n+i}^{2n+j} = 1, \delta_{3n+i}^{3n+j} = 1.$

And therefore

$$
\begin{aligned}
&ab\left[Z^i\frac{\partial^2 L}{\partial x_j\partial x_{3n+i}} + Z^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{3n+i}} + Z^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{3n+i}} + Z^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{3n+i}}\right]dx_i + ab(Z^i\frac{\partial^2 L}{\partial x_j\partial x_{2n+i}} \\
&+Z^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{2n+i}} + Z^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{2n+i}} + Z^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{2n+i}})dx_{n+j} + ab(Z^i\frac{\partial^2 L}{\partial x_j\partial x_{n+i}} + Z^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{n+i}}
\end{aligned}
$$

$$+Z^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{n+i}}+Z^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{n+i}})dx_{2n+j}+ab(Z^{i}\frac{\partial^2 L}{\partial x_{j}\partial x_{i}}+Z^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{i}}+Z^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{i}}\quad +Z^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{i}})dx_{3n+i}+$$
$$\left[\frac{\partial L}{\partial x_j}dx_j+\frac{\partial L}{\partial x_{n+j}}dx_{n+j}+\frac{\partial L}{\partial x_{2n+j}}dx_{2n+j}+\frac{\partial L}{\partial x_{3n+j}}dx_{3n+j}\right]=0 \longrightarrow \tag{17}$$

In this equation can be concise manner

$$ab\sum_{a=0}^{3}Z^{an+i}\frac{\partial^2 L}{\partial x_{an+j}\partial x_{3n+i}}dx_i+\frac{\partial L}{\partial x_j}dx_j+ab\sum_{a=0}^{3}Z^{an+i}\frac{\partial^2 L}{\partial x_{an+j}\partial x_{2n+i}}dx_{n+i}+\frac{\partial L}{\partial x_{n+j}}dx_{n+j}$$
$$+ab\sum_{a=0}^{3}Z^{an+i}\frac{\partial^2 L}{\partial x_{an+j}\partial x_{n+i}}dx_{2n+i}+\frac{\partial L}{\partial x_{2n+j}}dx_{2n+j}+ab\sum_{a=0}^{3}Z^{an+i}\frac{\partial^2 L}{\partial x_{an+j}\partial x_{i}}dx_{3n+i}$$
$$+\frac{\partial L}{\partial x_{3n+j}}dx_{3n+j}=0 \qquad \longrightarrow \tag{18}$$

Then we have the equations

$$ab\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_i}\right)+\frac{\partial L}{\partial x_{3n+i}}=0 \quad , \quad ab\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{n+i}}\right)+\frac{\partial L}{\partial x_{2n+i}}=0$$
$$ab\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{2n+i}}\right)+\frac{\partial L}{\partial x_{n+i}}=0 \quad , \quad b\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{3n+i}}\right)+\frac{\partial L}{\partial x_{i}}=0 \qquad \rightarrow \tag{19}$$

Thus equations obtained in Eq (19) are called Euler-Lagrange equations constructed by means of Φ_L^H on a generalized-quaternionic $k\ddot{a}hler$ manifold (M, g, V) and then triples (M, Φ_L^H, Z) are named Mechanical systems on a generalized-quaternionic $k\ddot{a}hler$ manifold (M, g, V).

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