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RESEARCH ARTICLE

ON HYBRID CENSORED INVERTED EXPONENTIAL DISTRIBUTION

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ABSTRACT

The present study deals with the estimation procedure of the parameter of inverted exponential distribution based hybrid censored data. For estimation purpose, we consider both, Classical and Bayesian method of estimation. In classical set up, the maximum likelihood estimate of the parameter with its standard error and $(1-\delta)\times 100\%$ confidence interval are computed. Further, by assuming Jeffrey's invariant and gamma priors of the unknown parameter, Bayes estimate along with its posterior standard error and highest posterior density credible interval of the parameter are obtained. Markov Chain Monte Carlo technique such as Metropolis-Hastings algorithm has been utilized to generate simulated draws from the posterior density of the parameter. A real data set representing the survival times (in days) of guinea pigs injected with different doses of tubercle bacilli has been analyzed for illustrative purpose.

INTRODUCTION

The exponential distribution is most widely used distribution for lifetime data analysis, because of its simplicity and mathematical feasibility. However, in real world, we rarely come across the engineering systems which have constant hazard rate throughout their life duration. Therefore, it seems practical to assume hazard rate as a function of time, which led to the development of alternative model for lifetime data analysis. A number of lifetime models (like Weibull, gamma, generalized exponential etc.) have been proposed to model lifetime data that have monotonically increasing or decreasing hazard rate function, though, non-monotonicity of the hazard rate has also been observed in many situation. For example, in the course of the study of mortality associated with some of the diseases, the hazard rate initially increases with time and reaches after a peak after some finite period of times and then decline slowly (Singh *et al.,* 2012). In view of this, Inverted exponential distribution (IED) has been discussed as a lifetime model by Lin *et al*. (1989) in detail. They obtained Maximum Likelihood estimates (MLEs), confidence limits and uniformly minimum variance unbiased estimators for the parameters and reliability function of IED with complete sample. Dey (2007) estimate the parameter of IED by assuming the parameter involved in the model as a random variable (r.v.).

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Prakash (2009) discussed the properties of the Bayes estimator, Shrinkage estimator and Minimax estimator of the parameter under the SELF and GELF for the IED. He also presented the moments of the lower record value and the estimation of the parameter, based on a series of observed record values by the maximum likelihood and moment methods. Recently, Singh *et al.* (2012) propose Bayes estimators of the parameter and reliability function for the same under the general entropy loss function for complete, Type-I and Type-II censored samples. The inverted exponential distribution (IED) has the following probability distribution function (pdf)

$$f(x|\lambda)=\frac{\lambda}{x^2}e^{-\lambda/x} \quad ;x,\lambda>0 \quad \ldots (1)$$

Also the reliability and hazard function are given by

$$R(x|\lambda)=1-e^{-\lambda/x} \quad \ldots (2)$$

$$h(x|\lambda)=\frac{\lambda}{x^2}\frac{e^{-\lambda/x}}{1-e^{-\lambda/x}} \quad \ldots (3)$$

A key characteristic that distinguishes survival analysis from other areas in statistics is that survival data are usually censored. Censoring occurs when incomplete information is available about the survival times of some individual. Sometimes it is intentionally done, as the life testing experiments are very time consuming and expensive. So in order to stay competitive in market, the reliability practitioners are generally using censoring in lifetime experiment (Balakrishnan and Aggarwala 2000). In reliability literature, Type-I and Type-II censoring are the most commonly used censoring schemes. The mixture of Type-I and Type-II censoring scheme is known as hybrid censoring scheme. In this censoring scheme, n items are put on test and the test is terminated when the pre-chosen number R out of n items are failed or when a pre-decided time T on the test has been reached. In other words, we can say that the termination point of the test is $T^{\dagger} = \min\{X_{R:n}, T\}$. Note that complete sample situation as well as Type-I and Type-II right censoring scheme all are special case of hybrid censoring scheme. Epstein (1954) was the first to introduce hybrid censoring and it is quite applicable in reliability acceptance test in MIL-STD-781C (1977). After words, hybrid censoring scheme is used by many authors like Draper and Guttmen (1987); Chen and Bhattacharya (1988); Gupta and Kundu (1998) and Childs *et al*. (2003). Some recent studies based on hybrid censoring are Kundu (2007); Banerjee and Kundu (2008); Kundu and Pradhan (2009); Dube *et al*. (2011); Ganguly *et al*. (2012) and Gupta and Singh (2012). For more detail about hybrid censoring scheme one may refer to Balakrishnan and Kundu (2013). Throughout the article, they mention some open problems and suggest some possible future work for the benefit of readers interested in this area of research. To the best of our knowledge, the problem of parameter estimation of IED has yet not been considered under hybrid censored information.

In view of above considerations, the paper is organized as follows. In section 2, we describe the model under the assumption of hybrid censored data from IED. In section 3, we obtained the MLE of the unknown parameter. It is observed that the MLE is not obtained in closed form, so it is not possible to derive the exact distribution of the MLE. Therefore, we propose to use the asymptotic distribution of the MLE to construct the approximate confidence interval. Further, by assuming Jeffrey's invariant and gamma priors of the unknown parameter, Bayes estimate along with its posterior standard error and highest posterior density credible (HPD) interval of the parameter are obtained in section 4. Markov Chain Monte Carlo (MCMC) technique such as Metropolis-Hastings algorithm has been utilized to generate simulated draws from the posterior density of the parameter. In Section 5, a real data set representing the survival times (in days) of guinea pigs injected with different doses of tubercle bacilli has been analyzed for illustration purpose.

Model Description

Suppose $x_{1:n} < x_{2:n} < \ldots\ldots < x_{2:n}$ are ordered lifetimes observations of n independent units that are put to test under the same environmental conditions and test is terminated when a pre-chosen number R, out of n items have failed or a pre determined time T, on test has been reached. It is assumed that the failed item not replaced and at least one failure is observed during the experiment. Therefore, under this censoring scheme we have one of the following types of observations:

Case I: $\{x_{1:n} < \ldots\ldots\ldots < x_{R:n}\}$ if $x_{R:n} < T$

Case II: $\{x_{1:n} < \ldots\ldots\ldots < x_{d:n}\}$ if $1 \le d < R$ and $x_{d:n} < T < x_{d+1:n}$

For schematic representation of the hybrid censoring scheme refer to Kundu and Pradhan (2009). It may be mentioned that although we do not observe $x_{d+1:n}$, but $x_{d:n} < T < x_{d+1:n}$ means that the d^{th} failure took place before T and no failure took place between $x_{d:n}$ and T. Let the life time random variable X has inverted exponential distribution with parameter λ. Based on the observed data, the likelihood function is given by

Case I:

$$L\left(\underset{\sim}{x}|\lambda\right) = \lambda^R \prod_{i=1}^{R} \frac{1}{x_{i:n}^2} e^{-\lambda \sum_{i=1}^{R} \frac{1}{x_{i:n}}} \left(1 - e^{-\frac{\lambda}{x_{R:n}}}\right)^{n-R} \quad \ldots (1)$$

Case II:

$$L\left(\underset{\sim}{x}|\lambda\right) = \lambda^d \prod_{i=1}^{d} \frac{1}{x_{i:n}^2} e^{-\lambda \sum_{i=1}^{d} \frac{1}{x_{i:n}}} \left(1 - e^{-\frac{\lambda}{T}}\right)^{n-d} \quad \ldots (2)$$

Where $\underset{\sim}{x} = \left(x_{1:n}, x_{2:n}, \ldots\ldots\ldots\right)$

The combined likelihood for Case I and case II can be written as

$$L = L\left(\underset{\sim}{x}|\lambda\right) = \lambda^r \prod_{i=1}^{r} \frac{1}{x_{i:n}^2} e^{-\lambda \sum_{i=1}^{r} \frac{1}{x_{i:n}}} \left(1 - e^{-\frac{\lambda}{c}}\right)^{n-r} \quad \ldots (3)$$

Where,

$$r = \begin{cases} R & \text{for case I} \\ d & \text{for case II} \end{cases} \qquad c = \begin{cases} x_{R:n} & \text{for case I} \\ T & \text{for case II} \end{cases}$$

Maximum Likelihood Estimators

The log-likelihood function for equation (3) can be written as

$$\log L = r\log\lambda - 2\sum_{i=1}^{r} \log\left(x_{i:n}\right) - \lambda \sum_{i=1}^{r} \left(1/x_{i:n}\right) + (n-r)\log\left(1 - e^{-\lambda/c}\right) \quad \ldots (4)$$

The first and second derivative of equation in (4) with respect to λ is given by

$$\frac{\partial \log L}{\partial \lambda} = \frac{r}{\lambda} - \sum_{i=1}^{r} \left(1/x_{i:n}\right) + \frac{(n-r)e^{-\lambda/c}}{c\left(1 - e^{-\lambda/c}\right)} \quad \ldots (5)$$

$$\frac{\partial^2 \log L}{\partial\lambda^2} = -\frac{r}{\lambda^2} - \frac{(n-r)e^{-\lambda/c}}{c^2\left(1-e^{-\lambda/c}\right)^2} \quad \ldots (6)$$

The MLE of λ will be the solution of the following non-linear equation

$$\frac{r}{\lambda} - \sum_{i=1}^{r}\left(1/x_{i:n}\right) + \frac{(n-r)e^{-\lambda/c}}{c\left(1-e^{-\lambda/c}\right)} = 0 \quad \ldots (7)$$

The equation in (7) can be solved for $\hat{\lambda}$ by using some suitable numerical iterative procedure such as Newton-Raphson method. The observed Fisher's information is given by

$$I\left(\hat{\lambda}\right) = -\left.\frac{\partial^2 \log L}{\partial\lambda^2}\right|_{\lambda=\hat{\lambda}} \quad \ldots (8)$$

Also, the asymptotic variance of $\hat{\lambda}$ is given by

$$\operatorname{Var}\left(\hat{\lambda}\right) = \frac{1}{I\left(\hat{\lambda}\right)} \quad \ldots (9)$$

The sampling distribution of $\frac{\left(\hat{\lambda}-\lambda\right)}{\sqrt{\operatorname{Var}\left(\hat{\lambda}\right)}}$ can be approximated by a standard normal distribution. The large-sample $(1-\delta)\times 100$ % confidence interval for λ is given by $\left[\hat{\lambda}_L, \hat{\lambda}_U\right] = \hat{\lambda} \pm z_{\delta/2}\sqrt{\operatorname{Var}\left(\hat{\lambda}\right)}$.

Bayesian Estimation

In this section, we have conducted a Bayesian study by assuming the following independent gamma prior for λ;

$$g(\lambda)\ \alpha\ \lambda^{\alpha-1}e^{-\beta\lambda}\ ;\ \lambda > 0$$

Here the hyper parameters α and β assumed to be known real numbers. Based on the above prior assumption, the joint density function of the sample observations and λ becomes

$$L\left(\underset{\sim}{x}, \theta\right) \alpha\, \lambda^{r+\alpha-1} e^{-\lambda\left(\sum_{i=1}^{r}\left(1/x_{i:n}\right)+\beta\right)}\left(1-e^{-\lambda/c}\right)^{n-r} \quad \ldots (10)$$

Thus, the posterior density function of λ, given the data is given by

$$\pi\left(\lambda|\underset{\sim}{x}\right) = \frac{L\left(\underset{\sim}{x}|\lambda\right)g\left(\lambda|\alpha,\beta\right)}{\int_0^\infty L\left(\underset{\sim}{x}|\lambda\right)g\left(\lambda|\alpha,\beta\right)d\lambda} \quad \ldots (11)$$

Therefore, if $h(\lambda)$ is any function of λ, its Bayes estimate under the squared error loss function is given by

$$\hat{h}(\lambda) = E_{\lambda|data}\left[h(\lambda)\right] = \frac{\int_0^\infty h(\lambda)L\left(\underset{\sim}{x},\lambda\right)d\lambda}{\int_0^\infty L\left(\underset{\sim}{x},\lambda\right)d\lambda} \quad \ldots (12)$$

Since it is not possible to compute (11) and therefore (12) analytically. Therefore, we propose the one of the MCMC method such as Metropolis-Hastings algorithm to draw samples from the posterior density function and then to compute the Bayes estimate and HPD credible interval.

Metropolis-Hastings algorithm

Step-1: Start with any value satisfying target density $f\left(\lambda^{(0)}\right) > 0$.

Step-2: Using current $\lambda^{(0)}$ value, generate a proposal point λ_prop from the proposal density $q\left(\lambda^{(1)},\lambda^{(2)}\right) = P\left(\lambda^{(1)} \to \lambda^{(2)}\right)$ i.e., the probability of returning a value of $\lambda^{(2)}$ given a previous value of $\lambda^{(1)}$.

Step-3: Calculate the ratio at the proposal point λ_prop and current $\lambda^{(i-1)}$ as:

$$\varpi = \log\left[\frac{f\left(\lambda\_prop\right)q\left(\lambda\_prop,\lambda^{(i-1)}\right)}{f\left(\lambda^{(i-1)}\right)q\left(\lambda^{(i-1)},\lambda\_prop\right)}\right]$$

Step-4: Generate U from uniform on (0, 1) and take Z=log U.

Step-5: If $Z < \varpi$, accept the move i.e., λ_prop and set $\lambda^{(0)} = \lambda\_prop$ and return to Step-1. Otherwise reject it and return to Step-2.

Step-6: Repeat the above procedure N times and record the sequence of the parameter λ as $\lambda_1, \lambda_2, \ldots\ldots, \lambda_N$. Further, to remove the autocorrelation between the chains of λ, we only store every fifth generated value. Let the size of the sample we thus store is M=N/5.

Step-7: The Bayes estimate of λ and corresponding posterior variance is respectively taken as the mean and variance of the generated values of λ.

Stpe-8: Let $\lambda_{(1)} \le \lambda_{(2)} \le ...... \le \lambda_{(M)}$ denote the ordered value of $\lambda_{(1)}, \lambda_{(2)}, ..., \lambda_{(M)}$. Then, following Chen and Shao (1999), the $(1-\delta)\times 100$ % HPD interval for λ is

$$\left(\lambda_{\left(M+i^*\right)}, \lambda_{\left(M+i^*+[(1-\delta)(M-N)]\right)}\right)$$ where, i^* is so chosen that

$$\lambda_{\left(M+i^*+[(1-\lambda)(M-N)]\right)} - \lambda_{\left(M+i^*\right)} = \min_{N \le i \le (M-N)-[(1-\delta)(M-N)]} \left(\lambda_{(N+i+[(1-\delta)(M-N)])} - \lambda_{(N+i)}\right)$$

Note that in Step-1, we choose the ML estimate of λ as the starting value.

Data Analysis

In this section, we conduct a real data analysis for application purpose. Here we consider a real data representing the survival times (in days) of guinea pigs injected with different doses of tubercle bacilli, reported by Bjerkedal (1960). The regimen number is the common logarithm of the number of bacillary units in 0.5 ml. of challenge solution; i.e., regimen 6.6 corresponds to 4.0×10^6 bacillary units per 0.5 ml $\left(\log 4.0\times 10^6 = 6.6\right)$. Corresponding to regimen 6.6, the survival times of 72 observations are given below:

12, 15, 22, 24, 24, 32, 32, 33, 34, 38, 38, 43, 44, 48, 52, 53, 54, 54, 55, 56, 57, 58, 58, 59, 60, 60, 60, 60, 61, 62, 63, 65, 65, 67, 68, 70, 70, 72, 73, 75, 76, 76, 81, 83, 84, 85, 87, 91, 95, 96, 98, 99, 109, 110, 121, 127, 129, 131, 143, 146, 146, 175, 175, 211, 233, 258, 258, 263, 297, 341, 341, 376.

This data set was considered by Kundu and Howlader (2010) for Bayesian inference and prediction of the inverse Weibull distribution based on Type-II censored data. Recently, Singh *et al*. (2012) purposes Bayes estimators of the parameter and reliability function of inverted exponential distribution under the general entropy loss function based on complete, Type-I and Type-II censored samples for the same data set. For analyzing this data set with hybrid censoring, we have created three artificially hybrid censored data sets from the above complete (uncensored) data under the following censoring schemes:

Scheme 1: R = 54, T=100 Scheme 2: R = 36, T=75 Scheme 3: R = 25, T=60

Fig.-1: Plot of genrated λ versus iteration of MCMC algorithm

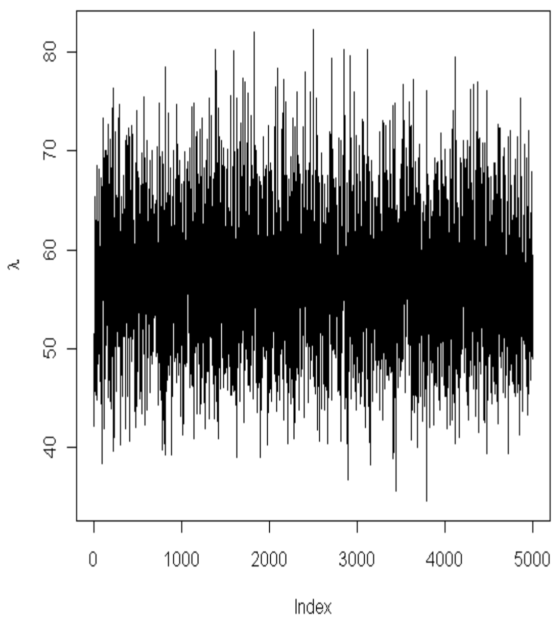


Fig.-2: Posterior densiy of λ

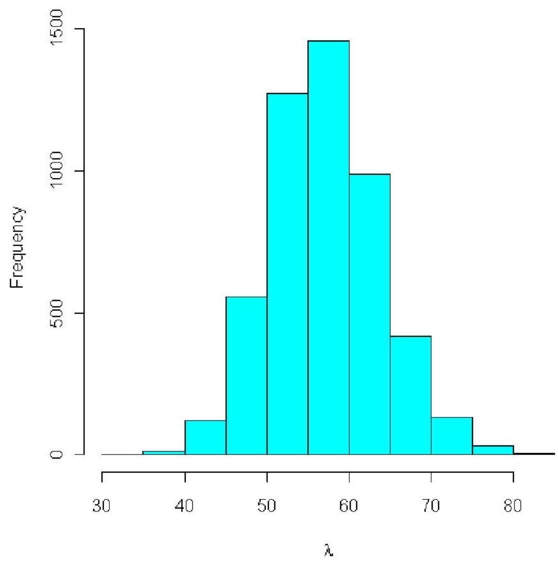


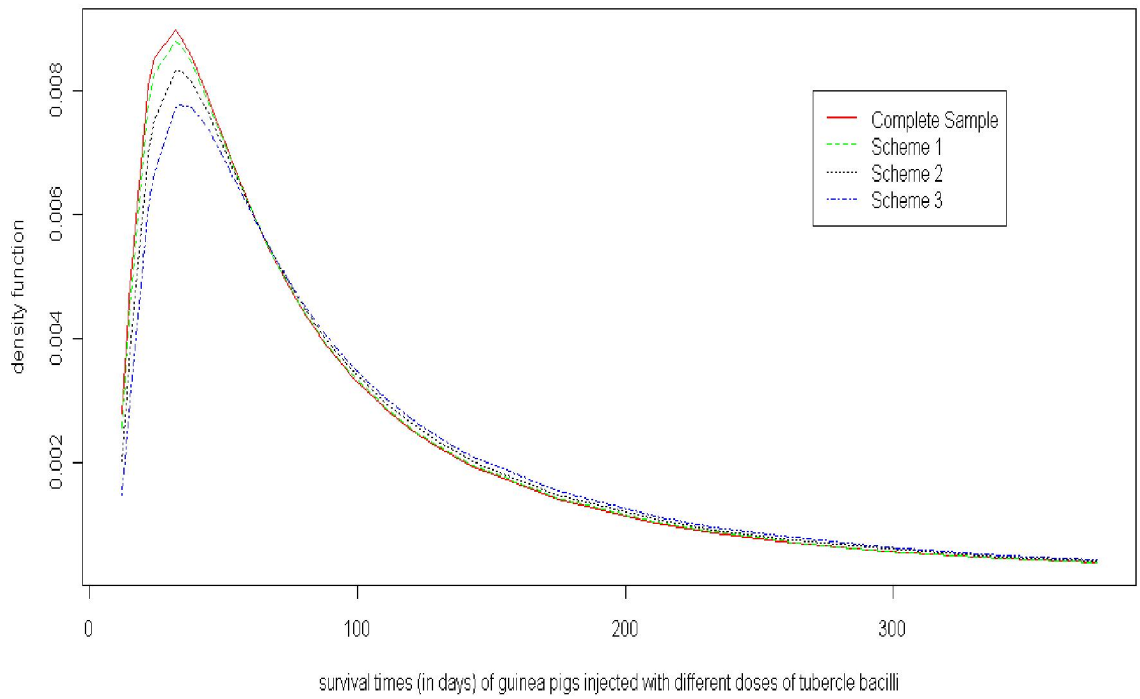
Table 1. Classical and Bayes estimates with their standard errors (S.E.)/ Posterior standard errors (PSE) for complete and hybrid censored data

Scheme/Estimates	ML estimate (SE)	Bayes estimates (PSE)	
		Jeffrey's Prior	Gamma Prior
Complete Sample	60.095 (7.083)	60.019 (7.044)	57.063 (6.692)
Scheme 1	61.354 (7.262)	61.238 (7.215)	58.274 (6.815)
Scheme 2	64.885 (7.792)	64.719 (7.712)	61.235 (7.311)
Scheme 3	69.572 (8.436)	69.543 (8.453)	64.963 (7.764)

Table 2. 95% Confidence/ HPD Intervals (CI) with their widths for complete and hybrid censored data

Scheme/Estimates	Confidence Interval {width}	HPD Intervals {width}	
		Jeffrey's Prior	Gamma Prior
Complete Sample	(46.214, 73.980) {27.766}	(46.622, 73.982) {27.359}	(44.223, 70.420) {26.196}
Scheme 1	(47.120, 75.589) {28.469}	(48.358, 75.280) {27.921}	(44.689, 71.129) {26.440}
Scheme 2	(49.692, 80.238) {30.545}	(49.243, 79.560) {30.317}	(46.317, 75.324) {28.378}
Scheme 3	(52.920, 86.225) {33.304}	(53.103, 86.124) {33.021}	(49.169, 79.063) {29.853}

Fig.3:- Different estimated density function based on MLE for complete, scheme1, scheme 2 and scheme 3 data sets



In all the cases, we have estimated the unknown parameter using the ML and Bayes methods of estimation. For obtaining MLE and 95% confidence interval, we have used nlm () function of R environment as the MLE of λ cannot be obtained in closed form. Bayes estimates of λ and HPD intervals are obtained using gamma and Jeffrey priors. Using Metropolis-Hastings algorithms; we generated 25,000 realizations of the parameter λ from the posterior density in (11). The convergence of the sequences of parameter for their stationary distributions has been checked through different starting values**.** The MCMC run of the parameter λ is plotted in Figure (Fig.) 1, which show fine mixing of the chains. We have also plot the posterior density of λ and found that it is symmetric (Fig. 2). For reducing the autocorrelation among the generated values of λ, we only record every 5^{th} generated values of each parameter. Initially, a strong autocorrelation is observed among the generated chain of λ. However, the serial correlation is minimized when we record only every 5^{th} generated outcomes. Bayes estimates of the parameter with gamma priors have been obtained by setting the values of prior's parameter as $\lambda = E(\lambda) = \beta / \alpha$ and put the value of prior's parameters as zero to obtain Bayes estimate with Jeffrey's prior. The results of the above three schemes have been summarized in Table 1-2. Note that, in the Tables 1-2, the entries in the bracket ( ) represents SEs/PSEs and that in the brackets () and { } respectively represent confidence/HPD interval and the widths of the interval. To see the consequence of censoring on the estimation of the unknown parameters, we have also plotted the four density functions based on MLE for complete, Scheme 1, Scheme 2 and Scheme 3 data sets in Fig. 3. For all the numerical computations, the programs are developed in R-environment. From the Table 1-2 and Fig. 2, we observed the following:

- Bayes estimation with gamma prior provides more precise estimates as compared to the MLEs (in terms of SE/PSE). Although Jeffrey priors perform similar to MLE even with the hybrid censored data.
- It is observed that the length of the HPD credible intervals based on informative priors are slightly shorter than the corresponding length of the HPD credible intervals based on non-informative priors, as expected.
- From Fig. 3, it is observed that the goodness of fit of the inverted exponential distribution is quite acceptable even with the hybrid censored data based on scheme 3. Although, estimated plot under all the censoring scheme could not estimate the upper tail properly because of the absence of information in that region. The loss of information increases respectively according to Scheme 1, Scheme 2 and Scheme 3, which is an obvious fact.

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