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RESEARCH ARTICLE

FINITE RANK OPERATORS AND FREDHOLM OPERATORS IN HILBERT SPACES

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ABSTRACT

In this paper we establish that every Fredholm operator F on a Hilbert space has a decomposition F=F+ K , where k is a finite rank operator. It is also shown that the product of two Fredholm operators can again be Fredholm.

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INTRODUCTION

Let H & B be two seperable Hilbert spaces.

Definition (1)

An operator K: H → B is said to have finite rank if Rank (K) c B is finite dimensional.

Remark

If K is a finite rank operator , then K is compact .In particular if either dim(H) < ∞ or dim(B) < ∞ then any bounded operator K: H→ B is finite rank and hence compact.

Definition (2)

A bounded operator F : H →B is Fredholm dim Nul (F) < ∞ ,dim co. Ker (F) < ∞ and Rank(F) is closed in B ,the index of F is the integer.

Index (F) = dim Nul (F) –dim Co Ker (F)
=dim Nul(F)-dim Nul (F*)

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Lemma (1)

Let M C H be a closed subspace and V C H be a finite dimensional subspace .Then M+V is closed as well .In particular ,if Co-dim (M)=dim(H/M)<∞ W C H is a subspace such that M C W ,then W in closed and Co-domain (W)<∞.
Lemma (2)

If K : H → B is a finite rank operator ,then there exists $\{\phi_n\}^{k}_{n=1}$ ⊃H and $\{\Psi_n\}^{k}_{n=1}$ ⊃B such that

(i) $K_x = \sum_{n=1}^{k} (x,\phi_n)\Psi_n$ for all x ε H
(ii) $K_y = \sum_{n=1}^{k} (y,\Psi_n)\phi_n$ for all y ε B

In Particular K* is still finite rank. For the next (3) & (4) .Let us assume B=H

(3) dim Nul (l+K) < ∞ ,
(4) dim Co Ker (l + K) < ∞ Rank (l + K) is closed and Rank (l + K) = Nul (l+ K*)^{l}

Theorem (1)

A bounded operator F: H→B is Fredholm if and only if there exists a bounded operator L: B→H such that (LF-I) & (FL-I) are both finite rank operators.

Proof

Suppose F: H→B is Fredholm , then F:Nul(F)$^{\perp}$→Rank(F) is a bijective bounded linear map between Hilbert spaces.

Let F^{-1}be the inverse of this map a bounded map by open mapping theorem. Let P:H→Rank (F) be orthogonal projection and set $L=F^{-1}P$,Then $(LF-I)=(F^{-1}PF-I)=F^{-1}F-I=-Q$ where Q is the orthogonal projection on to Nul (F).Similarly $(FL-I)=(FF^{-1}P-I)=-(I-P)$. Because (I-P) and Q finite rank projections and hence are finite rank. Therefore (LF-I) & (FL-I) are both finite rank operators. Conversely we shall first show that the operator L: B → H may be modified so that (LF-l) & (FL-l) are both finite rank operators. For this let G ≡(LF-I) & choose a finite rank approximation $G_1$to G such that $G=G_1+ \varepsilon$ where $||\varepsilon|| <$ 1.Define L : B→H to be the operator $L \equiv(1+\varepsilon)^{-1}L$ .Since $F=(1+\varepsilon\ )^{-1}L$.Since LF= $(1+\varepsilon)$ + $G_{1,}L$,$F=(I +(1+\varepsilon)^{-1}G_1=I+K$,where Kis a finite rank operator. Similarly there exists a bounded operator L_R:B→H and a finite-rank operator M_n such that $FL_R=I+M_R$. Note that $L_lFL_R=L_R+ M_RL_R$ and $L_lFL_R=L_l+ L_RM_R$ .There for $L_l-L_R=L_lM_R -K_lL_R=S$ is a finite rank operator. Therefore $FL_l=F(L_R+S)=I+M_R+FS$. So that there exists a bounded operator.

L^{-1}:B→H such that $(L^{-1}F-I)$ & (FL^{-1}-I) are both finite rank operator. We now assume that L is choosen such that $(LF-I)=G_{1,}(FL-I)=G_2$are finite rank ,clearly Nul (F) C Nul $(LF)=Nul(I+G_1)$

Rank (F) =Rank $(I+G_2)$

The theorem follows from Lemma (1) & (2)

Proposition (1)

If F: H→B is Fredholm then F* is Fredholm and index(F)=-index(F*)

Proof

Choose L: B→H such that both (LF-I) &(FL-I) are of finite rank. Then (F*L*-I) & (L*F*-I) are of finite ank which implies that F* is Fredholm. The assertion index (F)=-index (F*) follows directly from the definition (3).

Proposition (2)

Let F be a Fredholm operator & K be a finite rank operator from H → B and T be another Fredholm operator from B→X (where X is another Hilbert space)

Then (i) F+K is Fredholm and index (F)=index(F+K)

(ii)TF is Fredholm

Proof (1)

Given K : H→B, finite rank it is easily seen that F+K is still Fredholm .Indeed if L:B→H is a bounded operator such that $G_1=(LF-1)$ & $G_2=(FL-1)$ are both finite rank then L(F+K)-$1=G_2+KL$ are both finite rank. Hence F+K is Fredholm by Theorem(1).It is known that f(t)=index(F+ tk) is a continuous locally constant function of t ε R, and hence is constant .In particular, index(F+K)= f(t)= f(0)=index (F)

Proof (ii)

It is easily seen using theorem (1) that the product of two Fredholm operators is again Fredholm.

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