Available online at http://www.journalcra.com

International Journal of Current Research
Vol. 9, Issue, 02, pp.47109-47112, February, 2017

INTERNATIONAL JOURNAL OF CURRENT RESEARCH

ISSN: 0975-833X

RESEARCH ARTICLE

UNIQUE COMMON FIXED POINT FOR WEAKLY COMPATIBLE MAPS

[1]Rajesh Shrivastava and *,[2]Vipin Kumar Sharma

[1]Department of Mathematics, Excellence College, Bhopal, India
[2]Department of Mathematics, LNCT, Bhopal, India

ARTICLE INFO

Article History:
Received 20th November, 2016
Received in revised form
26th December, 2016
Accepted 16th January, 2017
Published online 28th February, 2017

Key words:

Fixed point, Complete metric space,
Weakly compatible maps.

ABSTRACT

This paper is investigated a common fixed point theorem in a metric space which generalizes the result of Jha (2007), using the weaker conditions such as Weakly compatible mappings and establish a unique common fixed point theorem for four self-mappings satisfying Lipschitz type contractive condition.

Citation: Rajesh Shrivastava and Vipin Kumar Sharma, 2017. "Unique common fixed point for weakly compatible maps", *International Journal of Current Research*, 9, (02), 47109-47112.

INTRODUCTION

In paper of Jha (2007), the most general of the common fixed point theorems pertain to four mappings, say A, B, S and T of a metric space (X, d), and use either a Banach type contractive condition of the form

$d(Ax, By) \leq h\, m(x, y),\ 0 \leq h < 1$ ... (1)

where $m(x, y) = \max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), (d(Sx, By) + d(Ax,Ty))/2\}$,

or, a Meir-Keeler type (ε, δ)-contractive condition of the form given $\varepsilon > 0$ there exists a $\delta > 0$ such that $\varepsilon \leq m(x, y) < \varepsilon + \delta \Rightarrow d(Ax, By) < \varepsilon$, …(2)

or, a φ-contractive condition of the form $d(Ax, By) \leq \varphi(m(x, y))$, ...(3)

involving a contractive gauge function $\varphi: R^+ \to R^+$ is such that $\varphi(t) < t$ for each $t > 0$. Clearly, condition (1) is a special case of both conditions (2) and (3). A φ-contractive condition (3) does not guarantee the existence of a fixed point unless some additional condition is assumed. Therefore, to ensure the existence of common fixed point under the contractive condition (3), the following conditions on the function φ have been introduced and used by various authors. (I)φ(t) is non decreasing and $t / (t - \varphi(t))$ is non increasing (Carbone *et al.,* 1989), (II) φ(t) is non-decreasing and $\lim n\, \varphi_n(t) = 0$ for each $t > 0$ (Jachymski, 1994), (III)φ is upper semi continuous (Boyd and Wong, 1969; Jachymski, 1994; Maiti and Pal, 1978; Pant 1998) or equivalently, (IV) φ is non decreasing and continuous from right (Park and Rhoades, 1981). It is now known (e.g., Jachymski, 1994; Pant *et al.,* 2001) that if any of the conditions (I), (II), (III) or (IV) is assumed on φ, then a φ-contractive condition (3) implies an analogous (ε, δ)-contractive condition (2) and both the contractive conditions hold simultaneously.

Similarly, a Meir- Keeler (1969) type (ε, δ)-contractive condition does not ensure the existence of a fixed point. The following example illustrates that an contractive condition of type (2) neither ensures the existence of a fixed point nor implies an analogous φ-contractive condition (3).

Example 1. (Pant *et al.,* 1998) Let X = (0, 2) and d be the Euclidean metric on X. Define $f : X \to X$ by $fx = (1+ x) /2$ if $x < 1$; $fx = 0$ if $x \geq 1$. Then, it satisfies contractive condition $\varepsilon \leq \max\{d(x, y), d(x, fx), d(y, fy), (d(x, fy) + d(y, fx))/2\} < \varepsilon + \delta \Rightarrow d(fx, fy) < \varepsilon$,

with $\delta(\varepsilon) = 1$ for $\varepsilon \geq 1$ and $\delta(\varepsilon) = 1 - \varepsilon$ for $\varepsilon < 1$ but f does not have a fixed point. Also f does not satisfy the contractive condition

**Corresponding author: Vipin Kumar Sharma,*
Department of Mathematics, LNCT, Bhopal, India.

$d(fx, fy) \leq \varphi(\max\{d(x, y), d(x, fx), d(y, fy), (d(x, fy) + d(y, fx))/2\})$,

since the desired function $\varphi(t)$ can not be defined at $t = 1$.

Hence, the two types of contractive conditions (2) and (3) are independent of each other. Thus, to ensure the existence of common fixed point under the contractive condition (2), the following conditions on the function δ have been introduced and used by various authors

(V) δ is non decreasing (Pant, 1993 & 1998)
(VI)δ is lower semi continuous (Jungck, 1986; Jungck *et al.*, 1993).

Jachymski (1994) has shown that the (ε, δ)-contractive condition (2) with a non decreasing δ implies a φ-contractive condition (3). Also, Pant *et al.* (2001) have shown that the (ε, δ)-contractive condition (2) with a lower semi continuous δ, implies a φ-contractive condition (3). Thus, we see that if additional conditions are assumed on δ then the (ε, δ)-contractive condition (2) implies an analogous φ-contractive condition (3) and both the contractive conditions hold simultaneously. It is thus clear that contractive conditions (2) and (3) hold simultaneously whenever (2) or (3)is assumed with additional condition on δ or φ respectively. It follows, therefore, that the known common fixed point theorems can be extended and generalized if instead of assuming one of the contractive condition (2) or (3) with additional conditions on δ and φ, we assume contractive condition weaker than the condition (2) together with the following Lipschitz type condition of the form

$$d(Ax, By) < k\Big(\frac{d(Ax, Sx)+d(Sx, By)}{1+d(Ax,Sx)d(Sx,By)} + \frac{d(By,Ty) + d(Ax,Ty)}{1+d(By,Ty)\, d(Ax,Ty)} + d(Sx,Ty) + d(Sx, Ty)d(Ax, Sx) + d(Ax, Sx)d(By, Ty) + d(Sx, Ty)d(By, Ty) + d(Ax, Sx)d(Sx, By) + d(By, Ty)d(Ax, Ty)\Big),$$ for $0 \leq k \leq 1/3$.

We prove a common fixed point theorem for four mappings adopting this approach in this paper. This gives a new approach of ensuring the existence of fixed points under an (ε, δ)-contractive condition consists of assuming additional conditions which are independent of theφ-contractive condition implied by (V) and (VI). As the fixed point theorem is established removing the assumption of continuity, relaxing the compatibility to the weak compatibility property and also replacing the completeness of the space, this result generalizes and improves various other similar results of fixed points. Two self-mappings A and S of a metric space (X, d) are called ***compatible*** (see Jungck (1986)) if, $\lim_{n\to\infty} d(ASx_n, SAx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some t in X. It is easy to see that compatible maps commute at their coincidence points.

Two self mappings A and S of a metric space (X, d) are called ***weakly compatible*** (see Jungck (1986)) if, they commute at coincidence points. That is, if Ax = Sx implies that ASx = SAx for x in X. To prove our theorem, we shall use the following Lemma of Jachymski (1994):

LEMMA (2.2 of Jachymski (1994)): Let A, B, S and T be self mappings of a metric space (X, d) such that $AX \subset TX$, $BX \subset SX$. Assume further that given $\varepsilon > 0$ there exists $\delta > 0$ such that for all x, y in X $\varepsilon < M(x, y) < \varepsilon + \delta \Rightarrow d(Ax, By) \leq \varepsilon$, (4) and $d(Ax, By) < M(x, y)$, whenever $M(x, y) > 0$ (5) where $M(x, y) = \max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), (d(Sx, By) + d(Ax,Ty))/2\}$. Then for each x_0 in X, the sequence $\{y_n\}$ in X defined by the rule$y_{2n} = Ax_{2n} = Tx_{2n+1}$; $y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$ is a Cauchy sequence. Jachymski (1994) has shown that contractive condition (2) implies (4) but contractive condition (4) does not imply the contractive condition (2).

RESULTS

Theorem 1. Let A, B, S and T be self mappings of a metric space (X, d) such that

(i) $AX \subset TX$, $BX \subset SX$,
(ii) Given $\varepsilon > 0$ there exists a $\delta > 0$ such that for all x, y in X, $\varepsilon < M(x, y) < \varepsilon + \delta \Rightarrow d(Ax, By) \leq \varepsilon$, and
(iii) $d(Ax, By) < k\Big(\frac{d(Ax, Sx)+d(Sx, By)}{1+d(Ax,Sx)d(Sx,By)} + \frac{d(By,Ty) + d(Ax,Ty)}{1+d(By,Ty)\, d(Ax,Ty)} + d(Sx, Ty) + d(Sx, Ty)d(Ax, Sx) + d(Ax, Sx)d(By, Ty) + d(Sx, Ty)d(By, Ty) + d(Ax, Sx)d(Sx, By) + d(By, Ty)d(Ax, Ty)\Big)$,

for $0 \leq k \leq 1/3$. If one of AX, BX, SX and TX is complete subspace of X and if the pairs (A,S) and (B, T) are weakly compatible, then A, B, S and T have unique common fixed point.

Proof. Let x_0 be an arbitrary point in X. Define sequences $\{x_n\}$ and $\{y_n\}$ in X given by the rule

$$y_{2n} = Ax_{2n} = Tx_{2n+1}\,;\ y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}. \qquad \ldots (6)$$

This can be done by virtue of (i). Since the contractive condition (ii) of this theorem implies the contractive conditions (4) and (5) of LEMMA (2.2 of Jachymski (3)), so using this LEMMA, we conclude that $\{y_n\}$ is a Cauchy sequence in X. Suppose that TX is a complete subspace of X, then the subsequence $y_{2n} = Tx_{2n+1}$ is a Cauchy sequence in TX and hence has a limit u. Let $v \in T^{-1}u$, then $Tv = u$. Since y_{2n} is convergent, so y_n is convergent to u and hence y_{2n+1} also converges to u. Now, setting $x = x_{2n}$ and $y = v$ in

(iii) we have,

$d(Ax_{2n}, Bv) < k\Big(\frac{d(Ax_{2n},Sx_{2n})+d(Sx_{2n},Bv)}{1+d(Bv,Tv)\, d(Ax_{2n},Tv)} + \frac{d(Bv,Tv)+d(Ax_{2n},Tv)}{1+d(Bv,Tv)d(Ax_{2n},Tv)} + d(Sx_{2n},Tv) + d(Sx_{2n}, Tv)d(Ax_{2n}, Sx_{2n}) + d(Ax_{2n}, Sx_{2n})d(Bv, Tv) + d(Sx_{2n}, Tv)d(Bv, Tv) + d(Ax_{2n}, Sx_{2n})d(Sx_{2n}, Bv) + d(Bv, Tv)d(Ax_{2n}, Tv)\Big)$

$d(Ax_{2n}, Bv) < k\,(d(Sx_{2n}, Tv) + d(Ax_{2n}, Sx_{2n}) + d(Bv, Tv) + d(Sx_{2n}, Bv) + d(Ax_{2n}, Tv) + d(Sx_{2n}, Tv)d(Ax_{2n}, Sx_{2n}) + d(Ax_{2n}, Sx_{2n})d(Bv, Tv) + d(Sx_{2n}, Tv)d(Bv, Tv) + d(Ax_{2n}, Sx_{2n})d(Sx_{2n}, Bv) + d(Bv, Tv)d(Ax_{2n}, Tv))$

Letting $n \to \infty$, we have $d(u, Bv) \leq 2k\, d(u, Bv)$, which implies that $Bv = u$. Also, since $BX \subset SX$, so $u = Bv$ implies that $u \in SX$. Let $w \in S^{-1}u$, then $Sw = u$. Setting $x = w$ and $y = x_{2n+1}$ in(iii), we get

$d(Aw,Bx_{2n}) < k\Big(d(Sw,Tx_{2n}) + \frac{d(Aw,Sw)+d(Sw,Bx_{2n})}{1+d(Aw,Sw)d(Sw,Bx_{2n})} + \frac{d(Bx_{2n},Tx_{2n})+ d(Aw,Tx_{2n})}{1+d(Bx_{2n},Tx_{2n})\, d(Aw,Tx_{2n})} + d(Sw, Tx_{2n})d(Aw, Sw) + d(Aw, Sw)d(By, Tx_{2n}) + d(Sw, Tx_{2n})d(Bx_{2n}, Tx_{2n}) + d(Aw, Sw)d(Sw, Bx_{2n}) + d(By, Tx_{2n})d(Aw, Ty_{2n})\Big)$

$d(Aw, Bx_{2n}) < k$ ($d(Sw, Tx_{2n})+d(Aw, Sw)+d(Bx_{2n}, Tx_{2n})+d(Sw, Bx_{2n}) + d(Aw, Tx_{2n})$
$+ d(Sw, Tx_{2n})d(Aw, Sw) + d(Aw, Sw)d(By, Tx_{2n})+d(Sw, Tx_{2n})d(Bx_{2n}, T_{2n})$
$+d(Aw, Sw)d(Sw, Bx_{2n}) + d(By, Tx_{2n})d(Aw, Ty_{2n}))$
and letting n tend to infinity, we get $d(Aw, u) \leq 2k\, d(u, Aw)$ which implies that u = Aw. This means that u = Tv = Bv = Aw = Sw. (7)
Now, since u = Tv = Bv, so by the weak compatibility of (B, T), it follows that BTv = TBv and so we get Bu = BTv = TBv = Tu.
Also, since u = Sw = Aw, so by the weak compatibility of (A, S), it follows that ASw = Saw and so we have Au = ASw = SAw = Su.
Thus, from (iii), we have
$d(Au, Bv) < k\,(d(Su, Tv)+\frac{d(Au,Su)+ d(Su,Bv)}{1+d(Au,Su)d(Su,Bv)}+\frac{d(Bv,Tv) + d(Au,Tv)}{1+d((Bv,Tv)\, d(Au,Tv)}$
$+ d(Su, Tv)\, d(Au, Su) + d(Au, Su)\, d(Bv, Tv) + d(Su, Tv)\, d(Bv, Tv)$
$+ d(Au, Su)\, d(Su, Bv) + d(Bv, Tv)\, d(Au, Tv))$,
$d(Au, Bv) < k\,(d(Su, Tv)+d(Au, Su) +d((Bv, Tv) + d(Su, Bv) + d(Au, Tv) +$
$d(Su, Tv)\, d(Au, Su) + d(Au, Su)\, d(Bv, Tv) + d(Su, Tv)\, d(Bv, Tv) + d(Au, Su)$
$d(Su, Bv) + d(Bv, Tv)\, d(Au, Tv))$,
that is, $d(u, Bu) < 3k\, d(u, Bu)$ which is a contradiction for $0 \leq k < 1/3$.

This implies that u =Bu. Similarly, using (iii), one can show that Au = u. Therefore, we have u = Bu = Tu = Au =Su. Hence, the point u is a common fixed point of A, B, S, and T. For uniquenes let v be a another fixed point of A, B, S, T.

Thus, from (iii), we have

$d(Au, Bv) < k(\frac{d(Au,Su)+d(Su,Bv)}{1+d(Au,Su)d(Su,Bv)}+\frac{d(Bv,Tv) + d(Au,Tv)}{1+d(Bv,Tv)\, d(Au,Tv)}+$
$d(Su,Tv) +d(Su, Tv)d(Au, Su)$
$+ d(Au, Su)d(Bv, Tv)+d(Su, Tv)d(Bv, Tv) +d(Au, Su)d(Su, Bv)$
$+ d(Bv, Tv)d(Au, Tv))$,
$d(Au, Bv) < k(d(Su, Bv) + d(Au, Tv) + d(Su, Tv) +d(Su, Tv)d(Au, Su)$
$+ d(Au, Su)d(Bv, Tv)+d(Su, Tv)d(Bv, Tv) +d(Au, Su)d(Su, Bv)$
$+ d(Bv, Tv)d(Au, Tv))$,
$d(u, v) < 3k\, d(u, v)$ which is a contradiction for $0 \leq k < 1/3$.This implies that u =v.

If we assume SX is complete, then the argument analogue to the previous completeness argument proves the theorem. If AX is complete, then $u \in AX \subset TX$. Similarly, if BX is complete, then $u \in BX \subset SX$. So, the theorem is established. The uniqueness of the common fixed point follows easily from condition (iii).This completely establishes the theorem.

We now give an example to illustrate the above theorem.

Example 2. Let X = (Carbone *et al*., 1989) and d be the usual metric on X. Define A, B, S, T : X → X as follows: Ax = 2 for each x; Sx = x if $x \leq 8$, Sx = 8 if $8 < x < 14$, Sx = (x +10)/3 if $14 \leq x \leq 17$

And Sx = (x+7)/3 if x >17; Tx = 2 if x = 2 or > 6, Tx = 12 + x if 2 < x< 4, Tx = 9 + x if $4 \leq x < 5$

And Tx = 8 if $5 \leq x \leq 6$; Bx = 2 if x < 4 or x > 6, Bx = 3 + x if $4 \leq x < 5$, Bx = 2 + x if $5 \leq x \leq 6$.

Then A, B, S and T satisfy all the conditions of the above theorem and have a unique common fixed point x = 2. Being compatible mappings, all A, B, S and T are weakly compatible mappings. It can be seen in this example that A, B, S and T satisfy the condition (4) when $\delta(\varepsilon) = 14 - \varepsilon$ if $\varepsilon \geq 6$ and $\delta(\varepsilon) = 6 - \varepsilon$ if $\varepsilon < 6$. It may also be noted that the mappings A, B, S and T do not satisfy the contractive condition (2). To see this, we can take x >17 and$5 \leq y \leq 6$, then we have $5 \leq d(Ax, By) \leq 6$ whereas $6 < M(x, y) < 8$. Thus the contractive condition (4) is satisfied but not (2) when x >17 and $5 \leq y \leq 6$. Also we see that $\delta(\varepsilon)$ is neither non decreasing nor lower semi continuous. However, A, B, S and T do not satisfy the contractive condition $d(Ax, By) \leq \varphi(M(x, y))$ since the required condition φ does not satisfy $\varphi(t) < t$ at t = 6. To verify this, we can take $8 < x \leq 17$ and $4 \leq y < 5$ then M(x, y) = 6 and d(Ax, By) → 6 as y → 5. Hence we see that the present example does not satisfy the condition of any previously known common fixed point theorem for continuous mappings since neither the mappings satisfy a φ-contractive condition nor δ is lower semi neither continuous nor non-decreasing.

Remarks

Pant (2003) has shown that condition (iii) of the above Theorem 1 is independent of φ-contractive conditions. Our result extends the results of Jha *et al.* (2005 & 2003), Jha and Pant (2003), Pant and Jha (2002) and Pant (2003) and gives a new generalization of Meir-Keeler type common fixed point theorem. Further, as various assumptions either on φ or on δ have been considered to ensure the existence of common fixed points under contractive conditions, so this Theorem 1 improves the results of Popa (2005).

REFERENCES

Boyd, D.W. and J.S. Wong, 1969. On nonlinear contractions, *Proc. Amer. Math. Soc*., 20, 458-464.

Carbone, A., B.E. Rhoades and S.P. Singh, 1989. A fixed point theorem for generalized contraction map, *Indian J. Pure Appl. Math*., 20, 543 - 548.

Jachymski, J. 1994. Common fixed point theorems for some families of mappings, *Indian J.Pure Appl. Math*., 25, 925 - 937.

Jha, K. 2007. Common fixed point theorems for weakly compatible maps in metric space, Kathmandu university *Journal of Science, Engineering and Technology*, 4.

Jha, K. and R.P. Pant, 2003. A generalization of a Meir-Keeler type common fixed point theorem for compatible maps, *Varahmihr J. Math. Sci*., 3, 27 – 34.

Jha, K., R.P. Pant and S.L. Singh, 2003. Common fixed points for compatible mappings inmetric space, Radovi Matematicki, 12, 107 – 114.

Jha, K., R.P. Pant and S.L. Singh, 2005. On the existence of common fixed points for compatible mappings, *Punjab Univ. J. Math*., 37, 39 – 48.

Jungck, G. 1986. Compatible mappings and common fixed points, *Internat. J. Math. Math.Sci*., 9, 771 - 779.

Jungck, G., K.B. Moon, S. Park and B.E. Rhoades, 1993. On generalizations of the Meir-Keeler type contractive maps: Corrections, *J. Math. Anal.Appl*., 180, 221 -222.

Maiti, M. and T.K. Pal, 1978. Generalization of two fixed point theorems, *Bull. Cal. Math.Soc*., 70, 57 - 61.

Meir, A. and E. Keeler, 1969. A theorem on contraction mappings, *J. Math. Anal. Appl*., 28, 326 - 329.

Pant, R.P. 1993. Common fixed points of weakly commuting mappings, Math. Student, 62, 97 - 102.

Pant, R.P. 1998. Common fixed points of contractive maps, J. *Math. Anal. Appl*., 226, 251 – 258.

Pant, R.P. 2003. Meir – Keeler type fixed point theorems and dynamics of functions, *Demonstratio Math*.,36(1)199–206.

Pant, R.P. and K. Jha, 2002. A generalization of Meir-Keeler type common fixed pointtheorem for four mappings, *J. Nat. Phys. Sci*., 16(1-2) 77 - 84.

Pant, R.P., P.C. Joshi and V. Gupta, 2001. A Meir-Keeler type fixed point theorem, *Indian J.Pure Appl. Math*., 32(6), 779 - 787.

Park, S. and B.E. Rhoades, 1981. Extension of some fixed point theorems of Hegedus and Kasahara, Math. Seminar Notes, 9, 113 - 118.

Popa, V. 2005. A generalization of Meir – Keeler type common fixed point theorem for four non-continuous mappings, Sarajevo J. Math., 1(13) 135 – 142.

****