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INTERNATIONAL JOURNAL OF CURRENT RESEARCH

International Journal of Current Research
Vol. 8, Issue, 10, pp.40459-40461, October, 2016

ISSN: 0975-833X

RESEARCH ARTICLE

BERNOULLI EQUATION IN FLUID FLOW

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ARTICLE INFO

Article History:
Received 22nd July, 2016
Received in revised form
19th August, 2016
Accepted 13th September, 2016
Published online 30th October, 2016

Key words:

Compressive, Conservation of energy, Steady flow, Streamline, Turbulent.

ABSTRACT

The Bernoulli equation is an approximate relation between pressure, velocity and elevation. It is valid in regions of steady, incompressible flow where net frictional forces are negligible. In this paper we discuss the first order differential equations such as linear and Bernoulli equation. From this we get some idea how differential equations are closely associated with physical applications to study real world problems which are described by first order differential equations. We introduce the equation of continuity and conservation of fluid flow, from which we derive Bernoulli equation.

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Citation: Thavamani, J. P. 2016. "Bernoulli equation in fluid flow", *International Journal of Current Research*, 8, (10), 40459-40461.

1. INTRODUCTION

Differential equations arise in many areas of science and technology, specifically whenever a deterministic relation involving some continuously varying quantities and their rate of change in space and/or time, expressed as derivatives is known or postulated. Many problems in engineering and science can be formulated in terms of differential equations. For differential equations of first order first degree we refer to (George F. Simmons second edition; Siddiq and Manchanda, 2006). The relationship between pressure and velocity in ideal fluids is described quantitatively by Bernoulli equation, named after its discoverer, the Swiss scientist Daniel Bernoulli (1700–1782). Bernoulli equation states that for an incompressible and in viscid fluid, the total mechanical energy of the fluid is constant. The total mechanical energy of a fluid exists in two forms, namely potential and kinetic. The kinetic energy of the fluid is stored in static pressure, ps, and dynamic pressure, $12\rho V_2$. Static pressure is simply the pressure at a given point in the fluid; dynamic pressure is the kinetic energy per unit volume of a fluid particle. Thus, a fluid will not have dynamic pressure unless it is moving. Therefore, if there is no change in potential energy along a streamline. For more details about fluid flow refer to (Lighthill, 1975). Bernoulli equation implies that the total energy along that streamline is constant and is a balance between static and dynamic pressure. The conservation of mass relation for a closed system undergoing a change is expressed as m = constant, which is a statement of the obvious that the mass of the system remains constant during a process.

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2. Differential Equation

The rate of change of a variable with respect to the other variable is called differentiation. An Equation involving derivatives of an unknown function is known as a differential equation. Such equations frequently arise when we wish to analyze mathematically, many of the phenomena that arise in nature or in various aspects of human endeavors. The following example illustrates how such differential equations may arise.
The distance x traversed in time t by a free particle of mass m, under the influence of a force f, is governed by Newton's second law of motion and is described by the differential equation

$$m\, d^2x/dt^2 = F \tag{2.1}$$

Here x is an unknown function of the independent variable t and F is a function of x and t. We consider the motion of a particle projected vertically upwards, from the surface of the earth. Assuming that the only force acting on the particle is the gravitational force of the earth, which is given by

$$F(x) = -\, GMm/x^2 \tag{2.2}$$

Where M is the mass of the earth, G is the universal gravitational constant and x is the distance of the particle from the centre of the earth, equation (2.1) takes the form

$$m\, d^2x/dt^2 = -\, GMm/x^2$$

In (2.2), the negative sign indicated that F acts opposite direction of x. Thus the acceleration of the particle is given by

$$d^2x/dt^2 = - GM/x^2 \tag{2.3}$$

Using chain rule of differentiation we can write

$$d^2x/dt^2 = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{dv}{dt} = \frac{dv}{dx}\cdot\frac{dx}{dt} = v\frac{dv}{dx}$$

where v is the velocity of the particle at time t. Now equation (2.3) becomes

$$v\frac{dv}{dx} = -\frac{GM}{x2} \tag{2.4}$$

But, at the earth's surface, the acceleration due to gravity is taken to be g, acting downwards. That is when x = R, $v\frac{dv}{dx} = d^2x/dt^2 = -g$, so that $GM = gR^2$ and equation (2.4) takes the form

$$v\frac{dv}{dx} = -\frac{gR2}{x2}$$

which is the differential equation in the unknown variable v considered as a function of x.

3. Bernoulli equation

Most of the time, linear differential equations are easiest to solve. In addition, many scientific phenomena are believed to be linear. That makes this class of equations a very important class. A generalized form of the linear equation that can be solved, is the Bernoulli equation

$$y' + p(x)y = q(x)y^{\alpha} \tag{3.1}$$

We Observe:

(i). When $\alpha = 0$, the Bernoulli equation becomes $y' + p(x)y = q(x)$, which is linear.
(ii). When $\alpha = 1$, the Bernoulli equation becomes $y' + p(x)y = q(x)y$ That is $y' + [p(x) - q(x)]y = 0$, which is both separable and linear.
(iii). When $\alpha \neq 0$ and $\alpha \neq 1$, the Bernoulli equation is non-linear.

Theorem

The Bernoulli equation $y' + p(x)y = q(x)y^{\alpha}$ can be solved by the change of variable $v = y^{1-\alpha}$

Proof

Assume that $\alpha \neq 0$ and $\alpha \neq 1$, because when $\alpha = 0$ and $\alpha = 1$, the given equation becomes linear and it is easy to solve.

Step:1 $v = y^{1-\alpha}$, the change of variable implies

$$\begin{aligned} v' &= (1 - \alpha)\, y^{-\alpha}\, y' \\ &= (1 - \alpha)\, y^{-\alpha}\, [-p(x)y + q(x)y^{\alpha}] \quad \text{[from equation (3.1)]} \\ &= (1 - \alpha)\, (-p\, y^{1-\alpha} + q) \\ &= (1 - \alpha)\, [-p(x)v + q(x)] \quad [\text{since } v = y^{1-\alpha}] \end{aligned}$$

The given equation turns to be $v' + (1-\alpha)\, p(x)v = (1-\alpha)q(x)$ (3.2) which is linear.

Step:2 Introduce the integrating factor

$$I(x) = e^{(1-\alpha)\int p(x)dx}$$

Multiplying the given equation by the integrating factor I(x), we get

$$I(x)v' + I(x)(1 - \alpha)\, p(x)v = I(x)(1 - \alpha)q(x)$$

We write $Iv' + I(1 - \alpha)\, pv = I(1 - \alpha)q$

$$(Iv)' = I(1 - \alpha)q$$
$$Iv = (1 - \alpha)\int(qI)dx$$

Hence the solution of equation (3.2) is given by

$$\begin{aligned} v &= \frac{1-\alpha}{I}\int (qI)\, dx \\ &= (1 - \alpha)\, e^{-(1-\alpha)\int p(x)dx} \int(q e^{(1-\alpha)\int p(x)dx})\, dx \end{aligned}$$

For the solution to equation (3.1), express it in terms of y by using $v = y^{1-\alpha}$.

That is $y^{1-\alpha} = (1 - \alpha)\, e^{-(1-\alpha)\int p(x)dx} \int(q e^{(1-\alpha)\int p(x)dx})\, dx$
Therefore $y = \{(1 - \alpha)\, e^{-(1-\alpha)\int p(x)dx} \int(q e^{(1-\alpha)\int p(x)dx})\, dx\}^{1/1-\alpha}$

Let us define a fluid as a "substance as a liquid, gas or powder, that is capable of flowing and that changes its shape at steady rate when acted upon by a force". The flow in which each point occupied by fluid its velocity does not change in time is called stationary flow. If velocity vectors components of fluid elements are not the functions of the time, the flow is called non-linear. If the flow is smooth, such that neighboring layers of the fluid slide by each other smoothly, the flow is said to be streamline. In this kind of flow, each particle of the fluid follows a smooth path, called a streamline, and these paths do not cross over one another. Above a certain speed, the flow becomes turbulent. The fluid can be compressive or non-compressive. Liquids can be thinking of as non-compressive fluids. Gases can be easily compressed, but the gases flows, if only the gas do not change its density during the flow, can be thinking of as non-compressive. If we look at a fluid flowing under normal circumstances, a river for example, the conditions at one point vary as time passes then we have unsteady flow. If the flow velocity is same magnitude and direction at every point in the fluid it is said to be uniform. A steady flow is one in which the conditions may differ from point to point but do not change with time. In this paper, assume that the fluid is essentially incompressible and that the flow is steady.

4. Continuity Equation

Consider the steady flow of liquid through an enclosed tube or pipe. In such a pipe, the mass must be conserved, if we put mass m_1 into the pipe, then the same mass $m_2 = m_1$ must flow out of this pipe. Consider infinitely small portion of mass, dm, put in a time dt into the pipe. From mass conservation we write

$$dm_1 = dm_2 \tag{4.1}$$

Knowing the relation of mass to volume and density, this reveals:

$$\rho\, dV_1 = \rho\, dV_2 \tag{4.2}$$

Volumes that fall into the pipe in a time dt is equal V = Adx, where A is the area of cross-section of the pipe and dx is the thickness of the mass layer, pumped in time dt. Substituting this to above equation, keeping in mind the definition of velocity, we have

$$A_1dx_1 = A_2dx_2$$
$$A_1v_1 = A_2\, v_2 \tag{4.3}$$

This equation is called the continuity equation for a fluid.

5. Derivation of Bernoulli Equation in fluid flow

We start with an overview of conservation principle and conservation mass relation. This is followed by a various forms of mechanical energy and the efficiency of mechanical work devices. Then we derive the Bernoulli equation by applying Newton's second law to a fluid element along a streamline. Equation (4.3) tells that where the cross-sectional area is large the velocity is small, and where the area is small the velocity is large. This is reasonable and can be observed by looking at a river. A river flows slowly through a meadow where it is broad, but speeds up to torrential speed when passing through a narrow gorge. Bernoulli principle states that where the velocity of a fluid is high, the pressure is low. To derive Bernoulli equation, assume the flow is steady and laminar the fluid is incompressible.

For a mass element $dm = \rho Adx$, we can write

$$E = E_k + E_p + E_i \qquad (5.1)$$

where the energy of the fluid at some position in the pipe, is a sum of kinetic energy, potential gravitational energy and internal energy, is given by $E_k = dm\, v^2/2$; $E_p = dm\, gh$; $E_i = PdV$ (m is the mass, v is the velocity and h is the height above the datum)

The energy components have self-explanatory meaning; kinetic energy is defined as usual, gravitational potential energy. To put the mass element dm into the pipe, to overcome some pressure p exists in the pipe. This pressure generates a force $F = \rho A$ that resists the motion. Moving by dx, a work needs to be done on the fluid. Divide the energy equation (5.1) by dV to obtain the Bernoulli equation. Bernoulli equation states that the energy of a fluid does not change in the flow. Since no energy is put to the fluid anywhere else, than in its input. Thus we have $dm = \rho\, dV$, that

$$P + \frac{1}{2}\rho v^2 + \rho gh = \text{constant} \qquad (5.2)$$

Taking any two points in the pipe and evaluating equation (5.2) for both of them, we obtain

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho gh_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho gh_2 \qquad (5.3)$$

Equation (5.3) is called the Bernoulli equation in a moving non-compressible fluid.

Remark: If there is no flow, $v_1 = v_2 = 0$, then equation (5.3) reduces to

$$P_1 + \rho gh_1 = P_2 + \rho gh_2$$
$$P_2 - P_1 = -\rho g(h_2 - h_1) \qquad (5.4)$$

Equation (5.4) is called hydrostatic equation.

6. Conclusion

The Bernoulli equation is the most famous equation in fluid mechanics. Its significance is that when the velocity increases in a fluid stream, the pressure decreases and when the velocity decreases, the pressure increases.

Acknowledgement

The author is thankful to the reviewers for their valuable suggestions to enhance the quality of the article.

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