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STRONG LINE DOMINATION IN GRAPHS

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ABSTRACT

For any graph $G = (V, E)$, the Line graph $L(G)$ of a graph G is a graph whose set of vertices is the union of the set of edges of G in which two vertices are adjacent if and only if the corresponding edges of G are adjacent. A dominating set D of a graph $L(G)$ is a strong Line dominating set if every vertex in $\langle V[L(G)] - D\rangle$ is strongly dominated by at least one vertex in D. Strong Line domination number $\gamma_{SL}(G)$ of G is the minimum cardinality of strong Line dominating set of G. In this paper, we study graph theoretic properties of $\gamma_{SL}(G)$ and many bounds were obtain in terms of elements of G and its relationship with other domination parameters were found.

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INTRODUCTION

In this paper, all the graphs consider here are simple and finite. For any undefined terms or notation can be found in Harary (Harary, 1972). In general, we use $<X>$ to denote the subgraph induced by the set of vertices X and $N(v)$ and $N((v))$ denote open (closed) neighborhoods of a vertex v. Let $deg(v)$ is the degree of vertex v and as usual $\delta(G)(\Delta(G))$ is the minimum (maximum) degree. A vertex of degree one is called an end vertex and its neighbor is called a support vertex. The degree of an edge $e = uv$ of G is defined by $\deg(e) = \deg(u) + \deg(v)$ and $\delta'(G)(\Delta'(G))$ is the minimum (maximum) degree among the edges of G. The notation $\alpha_o(G)(\alpha_1(G))$ is the minimum number of vertices (edges) in vertex (edge) cover of G. The notation $\beta_o(G)(\beta_1(G))$ is the maximum cardinality of a vertex (edge) independent set in G. A set $S \subseteq V(G)$ is said to be a dominating set of G, if every vertex in $V - S$ is adjacent to some vertex in S. The minimum cardinality of vertices in such a set is called the domination number of G and is denoted by $\gamma(G)$. The concept of edge dominating sets were also studied by Mitchell and Hedetniemi in (Mitchell and Hedetniemi, 1977).

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An edge dominating set of G if every edge in $E - F$ is adjacent to at least one edge in F. Equivalently, a set F edges in G is called an edge dominating set of G if for every edge $e \in E - F$, there exists an edge $e_1 \in F$ such that e and e_1 have a vertex in common. The edge domination number $\gamma'(G)$ of graph G is the minimum cardinality of an edge dominating set of G. A dominating set S is called the total dominating set, if for every vertex $v \in V$, there exists a vertex $u \in S$, $u \neq v$ such that u is adjacent to v. The total domination number of G is denoted by $\gamma_t(G)$ is the minimum cardinality of total dominating set of G. A dominating set $S \subseteq V(G)$ is a connected dominating set, if the induced subgraph $<S>$ has no isolated vertices. The connected domination number $\gamma_c(G)$ of G is the minimum cardinality of a connected dominating set of G. A dominating set $S \subseteq V(G)$ is restrained dominating set of G, if every vertex not in S is adjacent to a vertex in S and to a vertex in $V(G) - S$. The restrained domination number of a graph G is denoted by $\gamma_r(G)$ is the minimum cardinality of a restrained dominating set in G. The concept of restrained domination in graphs was introduced by Domke *et al.* (1999). A dominating set D of a graph $G = (V, E)$ is an independent dominating set if the induced subgraph $<D>$ has no edges.

The independent domination number $i(G)$ of a graph G is the minimum cardinality of an independent dominating set (Haynes *et al*., 1997; Robert B.Allan and Renu Laskar, 1978). The concept of a dominating set D of a graph G is a strong split dominating set if the induced subgraph $\langle V-D \rangle$ is totally disconnected with at least two vertices. The strong split domination number $\gamma_{ss}(G)$ of graph G is the minimum cardinality of a strong split dominating set of G. A dominating set D of a graph G is a global dominating set if D is also a dominating set of $\overline{G}$. The global domination number $\gamma_g(G)$ in the minimum cardinality of a global dominating set of G. This concept was introduced independently by Brigham and Dutton (Brigham and Dutton, 1990; Sampathkumar, 1989). The concept of Roman domination function (RDF) on a line graph $L(G)=(V',E')$ is a function $f:V' \to \{0,1,2\}$ satisfying the condition that every vertex u' for which $f(u')=0$ is adjacent to at least one vertex of v' for which $f(v')=2$ in $L(G)$. The weight of a Roman dominating function is the value $f(V')=\sum_{u' \in V'} f(u')$. The minimum weight of a Roman dominating function on a line graph $L(G)$ is called the Roman domination number of a graph $L(G)$ and is denoted by $\gamma_R(L(G))$ (see (9)). The concept of domination in graphs with its many were found in graph theory (Haynes *et al*., 1998; Haynes *et al*., 1999; Kulli *et al*., 1999; Panfarosh *et al*., 2014). Analogously, a dominating set D of a line $L(G)$ is a cototal dominating set if the induced subgraph $< V(L(G)) - D >$ has no isolated vertices. The cototal domination number $\gamma_{ct}(L(G))$ is the minimum cardinality of a cototal dominating set of $L(G)$(Panfarosh *et al*., 2014). The concept of Strong domination was introduced by Sampathkumar and Pushpa Latha in (1996) and well studied in (Muddebihal and Nawazoddin U. Patel, 2014; Muddebihal and Nawazoddin U. Patel, 2015; Muddebihal *et al*., 2015). Given two adjacent vertices u and v we say that u strongly dominates v if $\deg(u) \geq \deg(v)$. A set $D \subseteq V(G)$ is strong dominating set of G if very vertex in $V-D$ is strongly dominated by at least one vertex in D. The strong domination number $\gamma_s(G)$ is the minimum cardinality of a strong dominating set of G. A dominating set D of a graph $L(G)$ is a strong Line dominating set if every vertex in $\langle V[L(G)] - D \rangle$ is strongly dominated by at least one vertex in D. Strong Line domination number $\gamma_{SL}(G)$ of G is the minimum cardinality of strong Line dominating set of G. In this paper, many bounds on $\gamma_{SL}(G)$ were obtained in terms of elements of G but not the elements of $L(G)$. Also its relation with other domination parameters were established. We need the following theorem for our further results.

Theorem A(4): for any (p,q) graph G, $\gamma(G) = \lceil \frac{p}{2} \rceil$.

Main results

Theorem 1: For any non trivial (p,q) tree with $p \geq 3$ and m end vertices, then $\gamma_{SL}(G) \leq m$. Equality holds if $T = P_n, 4 \leq n \leq 7$.

Proof: Let $A = \{v_1, v_2, v_3, \ldots\ldots\ldots, v_n\} \subseteq V(T)$ be the set of all end vertices in T with $|A| = m$. Suppose $D \subseteq V - A$ be the set of all non end vertices then each block incident with the vertices of D gives a complete subgraph in $L(T)$.

If $\deg(u) \geq 2$, $u \in V(L(T))$, then $D' = \{u_1, u_2, u_3, \ldots\ldots\ldots, u_m\} \subseteq V[L(T)]$ such that $\deg(u_m) \geq \deg(u_k)$ $\forall u_k \in V(L(T)) - D'$ and $\forall u_m \in D$. Suppose $D'' = \{u_1, u_2, u_3, \ldots\ldots\ldots, u_i\}, 1 \leq i \leq m$ with $D' \subset V[L(T)] - D$ and $\deg(u_i) = \deg(u_j)$ $\forall u_j \in V[L(T)] - D'$. Then $\{D' \cup D''\}$ forms a minimal Strong dominating set of $L(T)$. Therefore, $|D' \cup D''| \leq m$ which gives $\gamma_{SL}(G) \leq m$. For equality if $P_n, 4 \leq n \leq 7$ holds, then for each P_4, P_5, P_6 and P_7 have $m = 2$. Since by Theorem A, $\gamma_{SL}(P_n) = 2 = m$; $4 \leq n \leq 7$. Then $\deg(u_i) \geq \deg(u_k)$ $\forall u_i \in D'$ and $\forall u_k \in V(L(G)) - D'$. Hence the equality.

Theorem 2: For any connected (p,q) graph G, $\gamma_{SL}(G) + \gamma(G) \leq P - 1$.

Proof: Let $R = \{v_1, v_2, v_3, \ldots\ldots\ldots, v_m\} \subseteq V(G)$ be the set of vertices with $\deg(v_j) \geq 2, \forall v_j \in R, 1 \leq j \leq m$. Further let there exists a set $R_1 \subseteq R$ of vertices with $diam(u,v) \geq 3, \forall u, v \in R_1$ which covers all the vertices in G. Clearly R_1 forms a dominating set of G. Otherwise if $diam(u,v) < 3$,then there exists at least one vertex $x \notin R_1$ such that $R' = R_1 \cup \{x\}$ form a minimal $\gamma - set$ of G. Now by definition of $L(G)$, let $H = \{u_1, u_2, u_3, \ldots\ldots\ldots, u_n\} \subseteq V[L(G)]$ be the set of vertices such that $\{u_i\} = \{e_i\} \in E(G)$, $1 \leq i \leq n$ where $\{e_i\}$ are incident with the vertices of R. Further let $D' \subseteq H$ be the set of vertices with $\deg(w) \geq 3$ for every $w \in D'$ such that $N[D'] = V(L(G))$ and if $\forall v_i \in V[L(G)] - D$. Then $\{D'\} \cup \{v_i\}$ forms a Strong line dominating set. Clearly $|\{D'\} \cup \{v_i\}| \cup |R'| = |V(G)| - 1$ and hence $\gamma_{SL}(G) + \gamma(G) \leq P - 1$.

Theorem 3: For any connected (p,q) graph G, $\gamma_{SL}(G) \leq p - \gamma_t(G)$.

Proof: Let $H = \{v_1, v_2, v_3, \ldots\ldots\ldots, v_m\}$ be the minimum set of vertices which covers all the vertices in G. Suppose $\deg(v_j) \geq 1, \forall v_j \in H_1, 1 \leq j \leq m$ in the subgraph $< H_1 >$ then H_1 forms a $\gamma_t(G) - set$ of G. Otherwise if $\deg(v_j) < 1$ then attach the vertices $w_i \in N(v_i)$ to make $\deg \geq 1$ such that $< H_1 \cup \{w_i\} >$ does not contains any isolated vertex. Clearly $H_1 \cup \{w_i\}$ forms a minimal total dominating set of G. Now in (G), let $A \subseteq V(L(G))$ be the set of vertices corresponding to the edges which are incident to the vertices of H in G. Let there exists a subset $D = \{u_1, u_2, u_3, \ldots\ldots\ldots, u_k\} \subseteq A$ of vertices with $\deg(u_i) \geq 3, 1 \leq i \leq k$ and $N[u_i] = V(L(G))$. Further $|\deg(u) - \deg(w)| \leq 2, \forall u \in D$ and $w \in V[L(G)] - D$ has at least one vertex in D.Clearly D forms a minimal Strong dominating set in $L(G)$. Therefore it follows that $|D| \leq |V(G)| - |H \cup \{w_i\}|$ and hence $\gamma_{SL}(G) \leq P - \gamma_t(G)$.

Theorem 4: For any connected (p,q) graph G, $\gamma_{SL}(G) + \gamma_c(G) + 2 \leq \alpha_o(G) + \beta_o(G) + \gamma(G)$

Proof: Let $A = \{v_1, v_2, v_3, \ldots\ldots\ldots, v_m\} \subseteq V(G)$ be the set of vertices with $\deg(v_j) \geq 2, \forall v_j \in A, 1 \leq j \leq m$ which are at distance at least two covers all the edges in G. Clearly $|A| = \alpha_o(G)$. Further if for any vertex $x \in A$, $N(x) \in V(G) - A$. Then A itself is an independent vertex set. Otherwise $A_1 \cup A_2$ where $A_1 \subseteq A$ and $A_2 \subseteq V(G) - A$ forms a maximum independent set of G with $|A_1 \cup A_2| = {}_0(G)$.

Now let $S = A' \cup A''$ where $A' \subseteq A$ and $A'' \subseteq V(G) - A$ be the minimal set of vertices which covers all the vertices in G. Clearly S forms a minimal γ - set of G. Suppose the subgraph $\langle S \rangle$ has only one component. Then S itself is a connected dominating set of G. Otherwise if the subgraph $\langle S \rangle$ has more than one component, then attach the minimum number of vertices $\{w_j\} \in V(G) - S$ where $\deg(w_j) \geq 2$, which are between the vertices of S such that $S_1 = S \cup \{w_j\}$ forms exactly one component in the subgraph $\langle S_1 \rangle$. Clearly S_1 forms a minimal γ_c- set of G. Let $D = \{u_1, u_2, u_3 \ldots, u_k\} \subseteq C$ where C is the set of vertices corresponding to the edges which are incident with the vertices of S in G. The minimal set of vertices with $N[D] = V(L(G))$ and $\forall u_k \in D$ has degree greater or equal to those vertices $u_j \in V(L(G)) - D$. Clearly D forms a Strong line dominating set in $L(G)$. Therefore $|D| \cup |S_1| + 2 \leq |A| \cup |A_1 \cup A_2| \cup |S|$ and hence $\gamma_{SL}(G) + \gamma_c(G) + 2 \leq \alpha_0(G) + \ _0(G) + \gamma(G)$.

Theorem 5: For any connected (p,q)- graph G, $\gamma_{SL}(G) + \gamma_r(G) \leq \alpha_1(G) + \ _1(G) + \beta'(G)$.

Proof: Let $A = \{e_1, e_2, \ldots, e_n\} \subseteq E(G)$ be the maximal set of edges with $N(e_i) \cap N(e_j) = e$ for every $e_i, e_j \in A$, $1 \leq i \leq n$, $1 \leq j \leq n$ and $e \in E(G) - A$. Clearly A forms a maximal independent edge set in G. Suppose $B = \{v_1, v_2, \ldots, v_n\}$ be the set of vertices which are incident with the edges of A and if $|B| = P$. Then A itself is an edge covering number. Otherwise consider the minimum number of edges $\{e_m\} \subseteq E(G) - A$ such that $A_1 = A \cup \{e_m\}$ forms a minimal edge covering set of G. Let $C = \{v_1, v_2, \ldots, v_k\} \subseteq V(G)$ be the set of all end vertices. Then $S = C \cup C'$ where $C' \subseteq V(G) - C$ be the set of vertices covering all the vertices with $diam(u,v) \geq 3$, $\forall u \in C$, $v \in C'$ or for every vertex $w \in V(G) - S$ there exists at least one vertex $z \in V(G) - S$ where $z \cap w = \emptyset$ and $y \in S$. Clearly S forms a minimal γ_r- set of G. Suppose $C = \emptyset$. Then S itself forms a minimal γ_r $-set$ of G. Let $D = \{u_1, u_2, \ldots, u_k\} \subseteq V(L(G))$ be the minimum set of vertices with $N[u_j] = V(L(G))$ for every $u_j \in D$, $1 \leq j \leq k$. If $\forall v_i \in V(L(G))$ has degree at most 2 and $v_i \in V[L(G)] - D$ then $\{D\} \cup \{v_i\}$ forms a strong line dominating set. Hence $|\{D\} \cup \{v_i\}| = \gamma_{SL}(G)$. Since for any graph G there exists at least one edge e with $|\deg(e)| = \beta'(G)$. Thus $|\{D\} \cup \{v_i\}| \cup |S| \leq |A_1| \cup |A| \cup |\deg(e)|$

There fore $\gamma_{SL}(G) + \gamma_r(G) \leq \alpha_1(G) + \ _1(G) + \beta'(G)$. The following theorem relates the Strong line domination number and Roman domination number of T.

Theorem 6: For any non trivial tree T with $p \geq 3$, then $\gamma_{SL}(T) \leq \gamma_R(T) - \Delta(T) + 1$.

Proof: Let $f: V(T) \to \{0,1,2\}$ and partition the vertex set $V(T)$ into (V_0, V_1, V_2) induced by f with $|V_i| = n_i$ for $i = 0,1,2$. Suppose the set V_2 dominates V_0. Then $S = V_1 \cup V_2$ forms a minimal Roman dominating set of T. Further let $A = \{v_1, v_2, \ldots, v_i\} \subseteq V(L(T))$ be the set of vertices with $\deg(v_j) \geq 3$. Suppose there exists a vertex set $D \subseteq A$ with $N[D] = V(L(T))$ and if $|\deg(x) - \deg(y)| \leq 2$, $\forall x \in D$, 0 $y \in V(L(T)) - D$. Then D forms a Strong line dominating set in $L(T)$. Otherwise there exists at least one vertex $\{w\} \subseteq A$ where $\{w\} \notin D$ such that $D \cup \{w\}$ forms a minimal $\gamma_{SL} - set$ in $L(T)$. Since for any tree G there exists at least one vertex $v \in V(T)$ of maximum degree $\Delta(T)$, then $|D \cup \{w\}| \leq |S| - |\deg(v)| + 1$. Clearly, $\gamma_{SL}(T) \leq \gamma_R(T) - \Delta(T) + 1$.

Theorem 7: A Strong line dominating set $D \subseteq V(L(G))$ is minimal if and only if for each vertex $x \in D$, one of the following condition holds.

a) There exists a vertex $y \in V(L(G)) - D$ such that $N(y) \cap D = \{x\}$
b) x is an isolated vertex in $\langle D \rangle$.
c) $\langle (V(L(G)) - D) \cup \{x\} \rangle$ is connected.

Proof: Suppose D is a minimal Strong line dominating set of G and there exists a vertex $x \in D$ such that x does not hold any of the above conditions. Then for some vertex v the set $D_1 = D - \{v\}$ forms a Strong line dominating set of G by the conditions (a) and (b). Also by (c), $\langle V(L(G)) - D \rangle$ is disconnected. This implies that D is a Strong line dominating set of G, a contradiction. Conversely, suppose for every vertex $x \in D$ one of the above statements hold. Further if D is not minimal. Then there exists a vertex $x \in D$ such that $D - \{x\}$ is a Strong line dominating set of G and there exists a vertex $y \in D - \{x\}$ such that y dominates x. That is $y \in N(x)$. Therefore x does not satisfy (a) and (b). Hence it must satisfy (c). Then there exists a vertex $y \in V(L(G)) - D$ and $N(y) \cap D = \{x\}$. Since $D - \{x\}$ is a Strong line dominating set of G, then there exists a vertex $z \in D - \{x\}$ and $z \in N(y)$. Therefore $w \in N(y) \cap D$ where $w \neq x$, a contradiction to the fact that $N(y) \cap D = \{x\}$ and $< V[(L(G)) - D] \cup \{x\} >$ is connected. Clearly D is a minimal Strong line dominating set of G.

Theorem 8: For any connected (p,q) graph G, $\gamma_{SL}(G) + \gamma_c(G) \leq diam(G) + \gamma(G) - 1$. Equality holds with $p \geq 3$.

Proof: Let $A \subseteq V(G)$ be the minimal set of vertices. Further, there exists an edge set $J \subseteq J'$ where J' is the set of edges which are incident with the vertices of A constituting the longest path in G such that $|J| = diam(G)$. Let $S' = \{v_1, v_2, \ldots, v_k\} \subseteq A'$ be the minimal set of vertices which covers all the vertices in G. Clearly S' forms a minimal dominating set of G. Suppose the subgraph $< S' >$ is connected. Then S' itself is a $\gamma_c - set$. Otherwise there exists at least one vertex $x \in V(G) - S'$ and $S'' = S' \cup \{x\}$ forms a minimal connected dominating set of G. Now in $L(G)$, let $F' = \{u_1, u_2, \ldots, u_n\} \subseteq V(L(G))$ be the set of $\{u_j\} = \{e_j\} \in E(G)$, $1 \leq j \leq n$ where $\{e_j\}$ are incident with the vertices of S'.

Further let $D \subseteq F'$ be the set of vertices with $N[D]=V(L(G))$ and $\forall u_k \in \langle V(L(G))-D \rangle$, $\deg(u_k) \le \deg(u_j)$ where $\forall u_j \in D$. Then D forms a Strong line dominating set of G. Otherwise there exists at least one vertex $\{u\} \in V(L(G))-D$ such that $\deg(u) > \deg(u_j)$, $\forall u_j \in D$. Clearly $D \cup \{u\}$ forms a minimal ¥$_{SL}-set$ of G. Thus $|D \cup \{u\}| \cup |S'| \le |J| \cup |S'| - 1$. Hence ¥$_{SL}(G)+$¥$_{c}(G) \le diam(G)+$¥$(G)-1$.

Theorem 9: For any non trivial tree T with $P \ge 3$ vertices and C number of cut vertices, then $\gamma_{SL}(T) \le C$.

Proof: Let $F' = \{v_1, v_2, \ldots, v_k\} \subseteq V(T)$ be the set of all cut vertices in T with $|F'| = C$. Further, let $A' = \{e_1, e_2, \ldots, e_k\}$ be the set of edges which are incident with the vertices of F'. Now by the definition of line graph, suppose $D = \{u_1, u_2, \ldots, u_j\} \subseteq A'$ be the set of vertices which covers all the vertices in $L(T)$. $\deg(u_k) \ge \deg(u_n)$ where $\forall u_k \in D$ and $u_n \in V(L(T) - D)$. Clearly D forms a minimal Strong line dominating set of $L(T)$, which gives $|D| \le |F'|$. Hence ¥$_{SL}(T) \le C$.

Theorem 10: For any connected (p,q) graph G, ¥$_{SL}(G) \le \left\lceil \frac{p}{2} \right\rceil$.

Proof: Let $D = \{v_1, v_2, \ldots, v_m\} \subseteq V(L(G))$ be the minimal Strong line dominating set of G. Suppose $|V(L(G))-D| = 0$. Then the result follows immediately. Further if $|V(L(G))-D| \ge 2$, then $V(L(G))-D$ contains at least two vertices such that $2n < p$. Hence ¥$_{SL}(G) = n < \lceil p/2 \rceil$.

Theorem 11: For any non trivial tree T and $T \ne K_{1,n}$ $n \ge 2$, then ¥$_{SL}(T) \le q - \Delta(T)$.

Proof: Let $B = \{v_1, v_2, \ldots, v_n\} \subseteq V(L(T))$ be the set of all vertices. Suppose there exists a set of vertices $B' = \{u_1, u_2, \ldots, u_m\} \subseteq V(L(T)) - B$ such that $dist(u_j, v_k) \ge 2$, $\forall u_j \in B'$, $v_k \in B$, $1 \le j \le m$, $1 \le k \le n$. Then $S = B \cup B'$ forms a Strong line dominating set of T. Otherwise if $B \not\subset V(L(T))$, then select the set of vertices $S = B'$ such that $N[S] = V(L(T))$ and the subgraph is disconnected. Clearly in any case S forms a minimal Strong line dominating set of T. Since for any tree T there exists at least one edge $e \in E(T)$ with $\deg(e) = \Delta(T)$. We obtain $|S| \le |E(T)| - \Delta(T)$. Therefore ¥$_{SL}(T) \le q - \Delta(T)$.

Theorem 12: For any acyclic (p,q) graph G, $\gamma_{SL}(G) \le i(G)$. Where $i(G)$ is an independent domination number G.

Proof: Suppose $A = \{v_1, v_2, v_3, \ldots \ldots \ldots, v_n\} \subseteq V(G)$ be the set of vertices which covers all the vertices in G. Further, if $\forall v_i \in A$, $deg\, v_i = 0$, then A itself is an independent dominating set of G. Otherwise $S = A' \cup I$, where $A' \subseteq A$ and $I \subseteq V(G) - A$ forms a minimal independent dominating set of G. Now let $B = \{v_1, v_2, \ldots, v_m\} \subseteq V(L(G))$ be the set of all vertices. Suppose there exists a set of vertices $B_1 = \{u_1, u_2, \ldots, u_n\} \subseteq V(L(G)) - B$ and $\deg(u_i) \ge \deg(v_j)$, $\forall u_i \in B_1$, $v_j \in B$, $1 \le i \le n$, $1 \le j \le m$. Then $D = B \cup B_1$ forms a Strong line dominating set of G. Otherwise if $B \not\subset V(L(G))$, then select the set of vertices $S = B_1$ such that $N[D] = V(L(G))$ and $\forall u_k \in \langle V(L(G))-D \rangle$, then $\deg(u_k) \le \deg(u_j)$ where $\forall u_j \in D$. Clearly D forms a Strong line dominating set of G. Otherwise there exists at least one vertex $\{u\} \in V(L(G)) - D$ such that $\deg(u) > \deg(u_j)$, $\forall u_j \in D$. Clearly $D \cup \{u\}$ forms a minimal ¥$_{SL}-set$ of G. Hence $|D \cup \{u\}| \le |V(G)|$ and clearly $\gamma_{SL}(G) \le i(G)$.

Theorem 13: For any connected (p,q) graph G, ¥$_{SL}(G)=1$ if and only if $L(G)$ has at least one vertices of degree $|V(L(G))|-1$.

Proof: To prove this result we consider the following two cases.

Case 1: Suppose $L(G)$ has exactly one vertex v, $\deg(v) = |V(L(G))| - 1$. Then in this case $D' = \{v\}$ is a minimal ¥$_{SL}-set$. If $D'' = \{u\} \in N(v)$ in $V(L(G)) - D'$ $\deg(u) \le V(L(G))-2$. Then there exists at least one vertex $w \notin N(u)$ in $L(G)$ such that $D_1 = D'' \cup \{w\}$ forms a Strong line dominating set in $L(G)$ a contradiction.

Case 2: Suppose $L(G)$ contains at least two vertices u and v with $\deg(u) = |V(L(G))| - 1 = \deg(v)$ and $v \notin N(u)$. Then $D' = \{u\}$ dominates all the vertices in $L(G)$. Since $\deg(u) = |V(L(G))| - 1$ and $|V(L(G))| - D' = V(L(G)) - \{u\}$. Hence $D_1 = \{v\} \cup V_1$, where $V_1 \subseteq V(L(G)) - D'$ forms a ¥$_{SL} - set$ again a contradiction. Conversely, suppose $\deg(u) = |V(L(G))| - 1 = \deg(v)$, u and v are adjacent to all the vertices in $L(G)$. Then $D_1 = \{v\} \in N(u)$ where $u \in D'$, $v \in V(L(G)) - D'$ and vice – versa. In any case we obtain $|D_1| = 1$.

Therefore $¥_{SL}(G)=1$.

Theorem 14: For any connected (p,q) graph $G, \gamma_{SL}(G) \le \gamma_{ss}(G)$.

Proof: let S' be a maximum independent set of vertices in G and $S' \subset S$ be the of all isolated vertices in G. Then $(V-S')\cup S'$ is a Strong split dominating set of G. Since for each vertex $v \in (V-S')\cup S''$ either v is an isolated vertex in $<(V-S')\cup S''>$ or there exists a vertex $u \in S'-S'$ and v is adjacent to $u, (V-S')\cup S'$ is minimal. Since S' is maximum $(V-S')\cup S'$ is minimum. Thus $|(V-S')\cup S''| = \gamma_{ss}(G)$.

Let $F=\{e_1,e_2,e_3,\dots,e_n\}$ be set of edges in G and $F \subseteq E(G)$. Then in $L(G)$, $D=\{v_1,v_2,v_3,\dots\dots,v_n\}$ which corresponds to $\forall e_i \in F$. Let $\deg(e_i)$, $\forall e_i \in F$ and $\deg(e_j)\ \forall e_j \in E(G)-F$ such that $\deg(e_i) \ge \deg(e_j)$. Suppose $D'=\{v_1,v_2,v_3,\dots\dots,v_i\} \subseteq D$ and $N[v_k]=V(L(G))$, $\forall u_k \in D'$, $1 \le k \le i$. Then D' forms a $\gamma - set$. It follows that $|D'| \le |(V-S')\cup S''|$. Hence $\gamma_{SL}(G) \le \gamma_{ss}(G)$.

Corollary: For a tree $T=K_{1,n}$ with $n \ge 2$ vertices $\gamma_{SL}(T)=(n+1)-(\gamma'(T)+1)$.

Theorem 15: For any connected (p,q) graph $G, \gamma_{SL}(G)+\gamma_{ct}(L(G))+\gamma_L(G) \le q+1+\gamma(G)$.

Proof: Let $S=\{v_1,v_2,\dots,v_n\} \subseteq V(G)$ be the set of vertices with $\deg(v_i) \ge 2$. Suppose exists a set $S_1 \subseteq S$ of vertices with $dist(u,v) \ge 3$, which covers all the vertices in G. Then S_1 forms a dominating set of G. Otherwise if $diam(u,v)<3$, then there exists at least one vertex $x \notin S_1$ such that $S'=S_1 \cup \{x\}$ forms a minimal $¥$ - set of G. Hence $|S'|=\gamma(G)$. Let $C_1=\{v_1,v_2,\dots,v_n\} \subseteq V(L(G))$ be the set of vertices with $dist(u,v) \ge 3$. Suppose there exists a set $D_1 \subseteq C_1$ which covers all the vertices in $L(G)$. Then D_1 itself is a line dominating set. If $dist(u,v)<3$ and $N[D_1] \ne V(L(G))$, then $D''=D_1 \cup \{w\}$, where $w \notin N[v]$, $v \in D_1$ forms a minimal dominating set of $L(G)$. Hence $|D_1 \cup \{w\}| = ¥_L(G)$. The edges which are incident with the vertices of S in G corresponds to the set of vertices $S''=\{v_1,v_2,\dots,v_m\} \subseteq V(L(G))$. Let F' be the set of vertices with $\deg(v)=1$, $\forall v \in F'$.

Suppose $I=\{v_1,v_2,\ \dots,v_j\} \subseteq S''$ be the set of vertices with $diam(a,b) \ge 3$, where $a \in F', b \in I$. Then $D=F' \cup I$ covers all the vertices in $L(G)$. Hence D forms a $¥_{ct}$ - set of $L(G)$. Otherwise there exists a vertex $z \in N(F') \cup N(I)$ and $D=F' \cup I \cup \{z\}$ forms a minimal cototal dominating set of $L(G)$. Hence $|D|=¥_{ct}(L(G))$. We consider $A=\{e_1,e_2,\dots,e_k\}$ be the set of all edges which are incident to the vertices of F'. Since $V(L(G))=E(G)$, then $D'=\{u_1,u_2,\dots,u_i\} \subseteq A$ be the set of vertices which covers all the vertices in $L(G)$. Clearly D' forms a minimal Strong line dominating set of $L(G)$. Therefore it implies that $|D'| \cup |D| \cup |D_1 \cup \{w\}| \le |E(G)| \cup |S'|+1$. Thus $\gamma_{SL}(G)+\gamma_{ct}(L(G))+\gamma_L(G) \le q+1+\gamma(G)$.

Theorem 16: For any connected (p,q) graph $G, \gamma_{SL}(G) \le diam(G)$.

Proof: Let $J'=\{e_1,e_2,\dots,e_n\} \subseteq E(G)$ be the minimal set of edges which constitute the longest path between any two distinct vertices $u,v \in V(G)$ with $dist(u,v)=diam(G)$. Let $H=\{u_1,u_2,u_3,\dots\dots\dots,u_n\} \subseteq V[L(G)]$ be the set of vertices such that $\{u_i\}=\{e_i\} \in E(G)$, $1 \le i \le n$, where $\{e_i\}$ are incident with the vertices of J'. Suppose $D \subseteq H$ be the set of vertices with $\deg(w) \ge 3$ for every $w \in D$ such that $N[D']=V(L(G))$ and $\forall v_i \in V[L(G)]-D$. Then $\{D\} \cup \{v_i\}$ forms a Strong line dominating set. It follows that $|\{D\} \cup \{v_i\}| \le diam(G)$. Hence $\gamma_{SL}(G) \le diam(G)$.

Theorem 17: For any connected (p,q) graph G, $¥_{SL}(G)+¥_R(L(G)) \le p+\Delta(G)$.

Proof: Let $f: V(L(G)) \to \{0,1,2\}$ and partition the vertex set $V(L(G))$ into (V_0,V_1,V_2) induced by f with $|V_i|=n_i$ for $i=0,1,2$. Suppose the set V_2 dominates V_0. Then $S=V_1 \cup V_2$ forms a minimal roman dominating set of $L(G)$. Further, let $F=\{v_1,v_2,\dots,v_k\} \subseteq V(L(G))$ be the set of vertices with $\deg(v_j) \ge 2$. Suppose there exists a vertex set $D \subseteq F$ with $N[D]=V(L(G))$ and if $|\deg(x)-\deg(y)| \le 1$, $\forall x \in D$, $y \in V(L(G))-D$. Then D forms a Strong line dominating set in $L(G)$. Otherwise there exists at least one vertex $\{w\} \subseteq F$ where $\{w\} \notin D$ such that $D \cup \{w\}$ forms a minimal $¥_{SL} - set$ in $L(G)$. Since for any graph G there exists at least one vertex $v \in V(G)$ of maximum degree $\Delta(G)$, it follows that $|D \cup \{w\}| \cup |S| \le p \cup |\deg(v)|$. Clearly $¥_{SL}(G)+¥_R(L(G)) \le p+\Delta(G)$.

Theorem 18: For any connected (p,q) graph G, $\gamma_{SL}(G) \le \gamma(G)+\gamma_L(G)$.

Proof: Suppose $C = \{v_1, v_2, v_3, \ldots\ldots, v_n\} \subseteq V(G)$ be the set of vertices with $\deg(v_j) \geq 2$. Then there exists a minimal set $S \subseteq C$ and $N[S] = V(G)$. Clearly S forms a dominating set of G. Let $C_1 = \{v_1, v_2, \ldots, v_n\} \subseteq V(L(G))$ be the corresponding to the set of vertices C with $dist(u,v) \geq 3$. Suppose there exists a set $D_1 \subseteq C_1$ which covers all the vertices in $L(G)$. Then D_1 itself is a line dominating set. Further if $dist(u,v) < 3$ and $N[D_1] \neq V(L(G))$, then $D = D_1 \cup \{w\}$, where $w \notin N[v]$, $v \in D_1$ forms a minimal dominating set of $L(G)$.Hence $|D| = \gamma_L(G)$. Let $H = \{u_1, u_2, u_3, \ldots\ldots, u_n\} \subseteq V[L(G)]$ be the set of vertices such that $\{u_i\} = \{e_i\} \in E(G)$, $1 \leq i \leq n$ where $\{e_i\}$ are incident with the vertices of C_1. Suppose $D' \subseteq H$ be the set of vertices with $\deg(w) \geq 3$ for every $w \in D'$ and $N[D'] = V(L(G))$ and $\forall v_i \in V[L(G)]$ has degree at most 2, and $v_i \in V[L(G)] - D'$. Then $\{D'\} \cup \{v_i\}$ forms a Strong line dominating set. It follows that $|D' \cup \{v_i\}| \leq |S| \cup |D|$ and hence $\gamma_{SL}(G) \leq \gamma(G) + \gamma_L(G)$.

Theorem 19: For any connected (p,q) graph G, $\gamma_{SL}(G) \leq \gamma_t(G) + \gamma(G)$.

Proof: Let $C' = \{v_1, v_2, \ldots, v_n\} \subseteq V(G)$ be the set of all non end vertices in G. Suppose $C'' \subseteq C'$ and $\forall v_i \in V(G) - C'$ are adjacent to at least one vertex of C'. Then C' forms a γ - set of G. Further, let $F = \{e_1, e_2, \ldots, e_k\}$ be the set of edges which are incident to the vertices of C'', and hence $|C''| = \gamma(G)$. Let $S \subseteq C''$ be the γ_t - set of G. By the minimality for every vertex $v \in S$, the induced subgraph $\langle S - v \rangle$ contains an isolated vertex. Let $S_1 = \{v : v \in S\}$ and A be the set of isolated vertices in $\langle S_1 \rangle$, $B = S_1 - A$. Further let C be the minimum set of vertices of $S - S_1$ and each vertex of A is adjacent to some vertex of C. Clearly $|C| \leq |A|$. Suppose $S' = S - \{S_1 \cup C\}$ and every $u_i v_i \in \langle S' \rangle$, $1 \leq i \leq k$, clearly $|S'| = \gamma_t(\langle S' \rangle)$. Then S' forms a minimal total dominating set of G. Let $H = \{u_1, u_2, u_3, \ldots\ldots, u_n\} \subseteq V[L(G)]$ be the set of vertices where $\{u_i\} = \{e_i\} \in E(G)$, $1 \leq i \leq n$, and $\{e_i\}$ are incident with the vertices of C. Further let $D' \subseteq H$ be the set of vertices with $\deg(w) \geq 3$ for every $w \in D'$ such that $N[D'] = V(L(G))$ and if $\forall v_i \in V[L(G)]$ has degree at most 2 and $v_i \in V[L(G)] - D$. Then $\{D'\} \cup \{v_i\}$ forms a Strong line dominating set. Clearly it follows that $|D' \cup \{v_i\}| \leq |C''| \cup |S'|$ and hence $\gamma_{SL}(G) \leq \gamma_t(G) + \gamma(G)$.

Theorem 20: For any connected (p,q) graph G, $\gamma_{SL}(G) \leq \gamma_g(G)$. Where $\gamma_g(G)$ is a global domination number of G.

Proof: Let $S = \{v_1, v_2, v_3, \ldots\ldots, v_n\} \subseteq V(G)$ be an independent set of G. Since G has no isolated vertices, $V - S$ is dominating set of G. Clearly for very vertex $v \in S$, $(V - S) \cup \{v\}$ is a global dominating set of G. Since $|(V-S) \cup \{v\}| = \gamma_g(G)$. Let $D = \{v_1, v_2, v_3, \ldots\ldots, v_n\} \subseteq V(L(G))$ be the minimal dominating set of $L(G)$ and $\deg(v_i) \geq 2\ \forall v_i \in D$ with $\deg(v_k) \leq 2\ \forall v_k \in V[L(G)] - D$. Then D is a Strong dominating set of $L(G)$. It follows that $|D| \leq |(V-S) \cup \{v\}|$ and hence $\gamma_{SL}(G) = \gamma_g(G)$.

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