Available online at http://www.journalcra.com

International Journal of Current Research
Vol. 8, Issue, 09, pp.37905-37910, September, 2016

INTERNATIONAL JOURNAL OF CURRENT RESEARCH

ISSN: 0975-833X

RESEARCH ARTICLE

PARAMETRIC PRECONDITIONED GAUSS – SEIDEL ITERATIVE METHOD

*Kumar, V. B., Vatti and Shouri Dominic

Department of Engineering Mathematics, Andhra University College of Engineering (A), Andhra University, Visakhapatnam – 530003, Andhra Pradesh, India

ARTICLE INFO

Article History:
Received 23rd June, 2016
Received in revised form
24th July, 2016
Accepted 08th August, 2016
Published online 20th September, 2016

Key words:
M-matrix,
Preconditioner,
Gauss-Seidel Splitting,
Convergence,
Iterative Methods.

ABSTRACT

In this paper, the Parametric Preconditioned Gauss-Seidel Iterative Method is developed by using the Preconditioned Gauss-Seidel Iterative Method and the spectral radius of these two methods are compared. A numerical example is given to illustrate the superiority of the new result.

Citation: Kumar, V.B., Vatti and Shouri Dominic, 2016. "Parametric preconditioned gauss – Seidel iterative method" *International Journal of Current Research*, 8, (09), **37**905-**37**910.

INTRODUCTION

We consider the system

$$A\underset{\sim}{X} = b \tag{1.1}$$

Where $A, \underset{\sim}{X}, b \in R^{n\times n}$ are known non-singular co-efficient matrix, unknown vector and known vector of the linear system (1.1) respectively. Without loss of generality, if we let $A = I - L - U$ where *I* is the identity matrix and –*L* and –*U* are the strictly lower triangular and upper triangular parts of *A* respectively. Then the well known AOR method iterative matrix is

$$T_{r,\mu} = (I - rL)^{-1}[(1-\mu)I + (\mu - r)L + \mu U] \tag{1.2}$$

where $0 \le r \le \mu \le 1$ with $\mu \ne 0$

and the Gauss-Seidel iterative matrix is

$$T_{G.S} = (I - L)^{-1}U \tag{1.3}$$

Corresponding author: Kumar, V.B.,
Department of Engineering Mathematics, Andhra University College of Engineering (A), Andhra University, Visakhapatnam – 530003, Andhra Pradesh, India.

By considering

$$S = (s_{ij})_{n\times n} = \begin{cases} -a_{in}, & i = 1, 2, ..., n-1,\ j = n \\ 0 & \text{for others} \end{cases}$$

the preconditioned linear system as given by He Honghao *et al*. [2009] is

$$(I+S)A\underset{\sim}{X} = (I+S)b \tag{1.4}$$

where,

$$(I+S)A = (I + D_1 - L - L_1 - U - U_1) \tag{1.5}$$

Here $D_1, L_1 \text{ and } U_1$ are the diagonal, strictly lower and strictly upper triangular parts of $S - SL$ respectively. And it can be easily seen that *SU* is always a null matrix.

If $a_{in}.a_{ni} \neq 1\ (i = 1, 2, ..., n-1),$ then $(I + D_1 - L - L_1)^{-1}$ exists and hence the Preconditioned Gauss-Seidel Iterative Method (PGSM) iterative matrix exists.

And, the PGSM iterative matrix is

$$\overline{T}_{G.S} = \overline{M}_{G.S}^{-1}.\overline{N}_{G.S}^{-1}[\overline{N}_{G.S}] = (I + D_1 - L - L_1)^{-1}(U + U_1) \tag{1.6}$$

Where

$$\overline{M}_{G.S} = I + D_1 - L - L_1 \tag{1.7}$$

$$\overline{N}_{G.S} = U + U_1 \tag{1.8}$$

we below mention some of the preliminaries, lemmas and theorems as given in He Honghao *et al*.[2009].

A matrix $A = (a_{ij}) \in R^{n\times n}$ is called nonnegative if $a_{ij} \geq 0,\ i,\ j = 1, 2, ..., n.$

A matrix $A = (a_{ij}) \in R^{n\times n}$ is called Z-matrix if $a_{ij} \leq 0,\ i \neq j$, and it is called an M-matrix if *A* is a Z-matrix and $A = sI - B,\ \ B \geq 0,\ \ s > \circ(B),$ where $\circ(B)$ denotes the spectral radius of B.

A splitting $B = M - N$ is called regular if $M^{-1} \geq 0 \text{ and } N \geq 0,$ weak regular if $M^{-1} \geq 0 \text{ and } M^{-1}N \geq 0.$

A matrix $A = (a_{ij}) \in R^{n\times n}$ is called an M-matrix if *A* is a Z-matrix and $A^{-1} \geq 0.$

A matrix *A* is irreducible if the direct graph associated to *A* is strongly connected.

Lemma 1.1: Let $A \geq 0$ be an irreducible matrix. Then

- *A* has a positive real eigen value equal to its spectral radius;
- To $\circ(A)$ there corresponds an eigen vector $\underset{\sim}{X} > 0$;
- $\circ(A)$ is a simple eigen value of *A*.

Lemma 1.2: Let $A \geq 0,\ a > 0,$ then $a \leq \circ(A)$ if $a\underset{\sim}{X} \leq A\underset{\sim}{X}$ and $\underset{\sim}{X} > 0;$ moreover, $a < \circ(A)$ if $a\underset{\sim}{X} < A\underset{\sim}{X}$ and $\underset{\sim}{X} > 0.$

Lemma 1.3: Let $A = M - N$ be regular of *A*, then $\circ(M^{-1}N) < 1$ if and only if *A* is a non-singular M-matrix.

Lemma 1.4: Let $A = (a_{ij}) \in R^{n\times n}$ be a non-singular and irreducible M-matrix, then $\circ(T_{r,\mu}) < 1, \quad 0 \le r \le \mu \le 1, (\mu \ne 0),$ where $T_{r,\mu}$ be the iterative matrix of AOR method given by (1.2).

Lemma 1.5: Let $A = (a_{ij}) \in R^{n\times n}$ be a non-singular and irreducible M-matrix with $0 < a_{in}a_{ni} < 1, \; i = 1, 2, ..., n-1.$ then $T_{r,\mu}, \quad 0 \le r \le \mu \le 1 (\mu \ne 0)$ is nonnegative and irreducible matrix, where $T_{r,\mu}$ is defined by (1.2).

Lemma 1.6: Let $A = (a_{ij}) \in R^{n\times n}$ be an M-matrix with $0 < a_{in}a_{ni} < 1, \quad i = 1, 2, ..., n-1, \; \underset{\sim}{X} \in R^n$ is n-dimensional vector, then $\overline{M}^{-1}_{G.S} S\underset{\sim}{X} > 0.$

Theorem 1.1: Let $A = (a_{ij}) \in R^{n\times n}$ is non-singular and irreducible M-matrix with $0 < a_{in}a_{ni} < 1, \; i = 1, 2, ..., n-1,$ then

(1) $\circ(\overline{T}_{G.S}) < \circ(T_{G.S}) \quad$ if $\circ(T_{G.S}) < 1.$

(2) $\circ(\overline{T}_{G.S}) = 1 \quad$ if $\circ(T_{G.S}) = 1.$

(3) $\circ(\overline{T}_{G.S}) > \circ(T_{G.S}) \quad$ if $\circ(T_{G.S}) > 1.$

Theorem 1.2: Let $A = (a_{ij}) \in R^{n\times n}$ is non-singular and irreducible M-matrix with $0 < a_{in}a_{ni} < 1,$ $i = 1, 2, ..., n-1, \underset{\sim}{X} \in R^n$ then $\circ(\overline{T}_{G.S}) < \circ(T_{r,\mu}) < 1, \; 0 \le r \le \mu \le 1 (\mu \ne 0)$

where $\overline{T}_{G.S}$ and $T_{r,\mu}$ are defined by (1.6) and (1.2) respectively.

In the section 2, we introduce Parametric Preconditioned Gauss-seidel Iterative Method (PPGSM) and compare the spectral radius of the iterative matrices of PPGSM and PGSM. Whereas in the concluding section, we present a numerical example.

Parametric Preconditioned Gauss – Seidel Iterative Method (PPGSM)

We consider the preconditioned linear system

$$\overline{A}\underset{\sim}{X} = \overline{b}$$

i.e.; $(I + \sim S)A\underset{\sim}{X} = (I + \sim S)b$ (2.1)

where $\sim \ne 0$ being a relaxation parameter

Here,

$$(I + \sim S)A = (I + \sim D_1 - L - \sim L_1 - U - \sim U_1) \tag{2.2}$$

where D_1, L_1 and U_1 are the diagonal, strictly lower and strictly upper triangular parts of $S - SL$ respectively.

The inverse of $(I + \sim D_1 - L - \sim L_1)$ exists if $a_{in}a_{ni} \ne 1 \; (i = 1, 2, ..., n), \sim \ne 0$ and hence PPGSM iteration matrix $\overline{T}_{\sim G.S}$ which is defined by

$$\overline{T}_{\sim G.S} = \overline{M}^{-1}_{\sim G.S} . \overline{N}_{\sim G.S} \tag{2.3}$$

Where $\overline{M}^{-1}_{\sim G.S} = (I + \sim D_1 - L - \sim L_1)$ (2.4) and

$$\overline{N}_{\sim G.S} = (U + \sim U_1) \tag{2.5}$$

We now, prove the following theorem by using the lemmas stated in section 1.

Theorem 2.1: Let $A = (\mathrm{a}_{ij}) \in R^{n\times n}$ is irreducible M-matrix and non-singular with

$0 < a_{in}a_{ni} < 1,\ i = 1, 2, ..., n-1,\ x \in R^n$.

Then

- $\circ(\overline{T}_{\sim G.S}) < \circ(\overline{T}_{G.S})$ if $\circ(T_{G.S}) < 1$
- $\circ(\overline{T}_{\sim G.S}) \le \circ(T_{G.S})$ if $\circ(T_{G.S}) \ge 1$.

Proof: There exists a positive vector $\underset{\sim}{X}$ such that

$$(I - L)^{-1}U\underset{\sim}{X} = \mathrm{a}\,\underset{\sim}{X} \tag{2.6}$$

since *A* being an irreducible M-matrix. Here 'a' being the spectral radius of the Gauss-Seidel iterative matrix i.e; $\circ(T_{G.S})$.
From (2.6), we have

$$\mathrm{a}(I - L)\underset{\sim}{X} = U\underset{\sim}{X} \tag{2.7}$$

$$\begin{aligned}
\overline{T}_{\alpha G.S} - \lambda\underset{\sim}{X} &= \overline{M}_{\alpha G.S.}^{-1}(U + \alpha U_1)\underset{\sim}{X} - \lambda\underset{\sim}{X} \\
&= \overline{M}_{\alpha G.S}^{-1}(U + \alpha U_1)\underset{\sim}{X} - \overline{M}_{\alpha G.S}^{-1}\overline{M}_{G.S}\lambda\underset{\sim}{X} \\
&= \overline{M}_{\alpha G.S}^{-1}[(U + \alpha U_1)\underset{\sim}{X} - \lambda(I + \alpha D_1 - L - \alpha L_1)]\underset{\sim}{X} \\
&= \overline{M}_{\alpha G.S}^{-1}[\lambda(I - L) + \alpha U_1 - \lambda + \lambda\alpha D_1 + \lambda L + \lambda\alpha L_1)]\underset{\sim}{X} \\
&= \overline{M}_{\alpha G.S}^{-1}[\lambda(I - L) - \lambda(I - L) + \alpha U_1 - \lambda\alpha D_1 + \lambda\alpha L_1)]\underset{\sim}{X} \\
&= \overline{M}_{\alpha G.S}^{-1}[\alpha(U_1 - \lambda D_1 + \lambda L_1)]\underset{\sim}{X} \\
&= \alpha\overline{M}_{\alpha G.S}^{-1}[U_1 - \lambda D_1 + \lambda L_1]\underset{\sim}{X} \\
&= \alpha\overline{M}_{\alpha G.S}^{-1}[U_1 - D_1 + L_1 - (\lambda - 1)D_1 + (\lambda - 1)L_1]\underset{\sim}{X} \\
&= \alpha\overline{M}_{\alpha G.S}^{-1}[SU + SL - S - (\lambda - 1)D_1 + (\lambda - 1)L_1]\underset{\sim}{X} \\
&= \alpha\overline{M}_{\alpha G.S}^{-1}[\lambda S(I - L) - S(I - L) - (\lambda - 1)D_1 + (\lambda - 1)L_1]\underset{\sim}{X} \\
&= \alpha\overline{M}_{\alpha G.S}^{-1}[(\lambda - 1)S(I - L) - (\lambda - 1)D_1 + (\lambda - 1)L_1]\underset{\sim}{X} \\
&= \alpha(\lambda - 1)\overline{M}_{\alpha G.S}^{-1}[L_1 - D_1]\underset{\sim}{X}
\end{aligned} \tag{2.8}$$

Let $y = [(I + \sim D_1) - (L + \sim L_1)]^{-1}(L_1 - D_1)\underset{\sim}{X}$ and $y' = (L_1 - D_1)\underset{\sim}{X}$. By the assumption $0 < a_{in}a_{ni} < 1$, $i = 1, 2, ..., n-1,$ we have $(L_1 - D_1)\underset{\sim}{X} = a_{in}a_{ni}\underset{\sim}{X}_i > 0,$ for $i = 1, 2, ..., n-1$. From which one can deduce that the first n-1 entries of y' are positive.

Since

$$[(I+\omega D_1)-(L+\omega L_1)]^{-1}=[I-(I+\omega D_1)^{-1}(L+\omega L_1)](I+\omega D_1)^{-1}$$
$$\geq [I-(I+\omega D_1)^{-1}(L+\omega L_1)] \geq (I-L)^{-1}$$
$$=(I+L+L^2+......) \geq I+L,$$

$y \geq (I+L)y' = y'+Ly'$ is obtained. Because of $a_{n-1n}a_{nn-1} > 0$, $(L)_{nn-1} > 0$ is obtained. Hence the n^{th} entry of Ly' is positive. So, we can deduce that *y* is positive.

If $0 < \alpha < 1$ and $\omega > 0$ then from (2.8) we have

$$\overline{T}_{\omega G.S} - \alpha \underset{\sim}{X} = \omega(\alpha - 1)y < 0 \tag{2.9}$$

by Lemma (1.2) and hence

$$\rho(\overline{T}_{\omega G.S}) < \rho(T_{G.S})$$

If $\alpha = 1$, then $\overline{T}_{\omega G.S} - \alpha \underset{\sim}{X} = \omega(\alpha - 1)y = 0$ (2.10)

Therefore, $\rho(\overline{T}_{\omega G.S}) = \rho(T_{G.S})$

If $\alpha > 1$ and $\omega < 0$, then $\overline{T}_{\omega G.S} - \alpha \underset{\sim}{X} = \omega(\alpha - 1)y < 0$ (2.11)

Therefore, $\rho(\overline{T}_{\omega G.S}) < \rho(T_{G.S})$ provided

$$\omega = \frac{1}{1-\alpha} \tag{2.12}$$

Finally, as long as 'ω' be fixed as in (2.12) we have for any 'α' in the ranges $0 < \alpha < 1$ and $\alpha > 1$,

$$\rho(\overline{T}_{\alpha G.S}) < \rho(T_{G-S})[T_{G.S}]$$

Hence the proof of the theorem is complete.

Numerical Examples

We considered the example given in He Honghao et.al. [2009] and tabulated the spectral radii of iterative matrices of AOR, PGSM and PPGSM in this paper. One can denote the following matrix '*A*' as non-singular and irreducible M-matrix.

$$A=\begin{pmatrix} 1 & -0.0709 & -0.0250 & -0.0240 \\ -0.0718 & 1 & -0.0472 & -0.0597 \\ -0.0735 & -0.0856 & 1 & -0.0783 \\ -0.1261 & -0.0107 & -0.0746 & 1 \end{pmatrix}$$

Table 1

(μ, r)	$\rho(T_{\mu,r})$
(0.6, 0.4)	0.4879
(0.8, 0.2)	0.3306
(0.9, 0.3)	0.2396
(1.0, 1.0)	0.0530

Table 2.

Iterative Matrix	Spectral Radius
$T_{G.S}$	0.0601
$\overline{T}_{G.S}$	0.0250
$\overline{T}_{\omega G.S}$ with ω =1.06394	0.0173

It can be seen from the above tabulated results that PPGSM with preconditioner $P = (I + \sim S)$ where $'\sim'$ is as given in (2.12), better than all the methods considered in this paper.

REFERENCES

Axelession, O. 1994. Iterative Solution Method [M]. Cambridge University Press, Cambridge.

Berban, R.J. 1994. Plemmons. Nonnegative Matrices in the Mathematical Sciences [J]. SIAM, Philadelphia, PA.

Elsner, L. 1989. Comparisons of Weak Regular Splitting and Multisplitting Methods [J]. Numer. Math., 56: 283-289, 1989.

Evans, D.J., Martins, M.M., Trigo, M.E. 2001. The AOR method for preconditioned liner [J], Journal of Computational and Applied Mathematics, 132: 461-466.

He Honghao, Yuan Dongjin, Hou Yi, Xu Jinqiu, 2009. Preconditioned Gauss-Seidel Iterative Method for Linear Systems. International Forum on Information Technology and Applications, 382-385, 2009.

Jiagan Hu., Iterative method for solving linear [M]. Science Press, Beijing, 1991.

Li – Ying Sun, 2005. Comparison theorem for the SOR iterative method [J]. Journal of Computational and Applied Mathematics, 181:336-341).

Varga, R.S. 1981. Matrix iterative Analysis [M], Prentice Hall, Eaglewood Cliff NJ, 1981.

Wen Li, W.W. Sun, 2000. Modified Gauss – Seidel methods and Jacobi type methods for Z – matrix [J]. Linear Algebra Appl., 317: 223-240.

****