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RESEARCH ARTICLE

OPERATIONAL CALCULUS ON FOURIER-STIELTJES TRANSFORM

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ABSTRACT

Fourier and Stieltjes transforms represent an important area of analysis and properties of it are more elegant. The Fourier transform is most significant in functional analysis, complex analysis, number theory, representation theory etc. Also, Fourier transform has applicable in many areas such as image processing, time series analysis, antenna design, radar system, human auditory system etc. In the same way, the Stieltjes transform is also a basic tool for analyzing the behavior of many important functions in mathematics and mathematical physics. As it is well known, the Stieltjes transform can be regarded as an eigenvalue moment generating function. The Stieltjes transform have many applications in many areas such as statistics, probability, moment problems, it is a key tool to derive information and communication theoretic performance measures for random vector channels; it can be used to express more intuitive performance measures of communication systems such as signal to interference, noise ratios and channel capacity etc. In this paper we present Operational calculus on Fourier-Stieltjes Transform.

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INTRODUCTION

The operational calculus of Integral transform essentially involves the replacement of the function under the study by other functions called transform, which are obtained from the original functions by certain rules. Also, we know some application of operational calculus of integral transform to integral equations, difference equation, fractional integrals, Fractional derivatives, summation of infinite series, evaluation of definite integrals, statics and problems of probability (Debnath and Batta, 2007; Khairnar *et al.*, 2012). Fourier transform (FT) is named in the honor of Joseph Fourier (1768-1830), one of greatest names in the history of mathematics and physics. Mathematically speaking, it is a linear operator that maps a functional space to another functions space and decomposes a function into another function of its frequency components. The formulae used to defined Fourier transform vary according to different authors. Fourier transform are use in many areas of geophysics such as image processing, time series analysis, and antenna design. It is use for solving linear partial differential equations (PDE). Some examples include: Poisson's equation for problems in gravity and magnetic; the biharmonic equation for problems in linear visco-elasticity and the diffusion equation for problems in heat conduction (Shubing Wang, 2007). Also, it is used in communication, data analysis and image processing etc. The generalized Stieltjes transform (GST) is an integral transform that depends on a parameter$p > 0$. It is a well known fact that the Generalized Stieltjes transform can be formulated as an iterated Laplace transform, and that therefore its inverse can be expressed as an iterated inverse Laplace transform. The conventional Fourier-Stieltjes transform of a complex valued smooth function $f(t,x)$ is defined by the convergent integral.

$$FS(s,p) = FS\{f(t,x)\} = \int_0^\infty \int_0^\infty f(t,x)\, e^{-ist}\, (x+y)^{-p}\, dt\, dx \qquad \ldots\ldots\ldots\ldots (1.1)$$

Where, t and x are positive real numbers.

The outline of the present paper

In this paper, we have defined the testing function spaces which are given in section 2. In section 3, the various properties for generalized Fourier-Stieltjes transform are proved. Lastly, conclusion is given. Notation and terminology as per Zemanian (1968).

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The Testing function space

The space is given by

$$FS_\alpha = \{\emptyset : \emptyset \in E_+ / \gamma_{k,p,l,q}\emptyset(t,x) = \sup_{I_1}\left|t^k(1+x)^p D_t^l (xD_x)^q \emptyset(t,x)\right| \le C_{plq}A^k k^{k\alpha}\} \quad \text{……… (2.1)}$$

Where the constant A and C_{plq} depend on the testing function space$\emptyset$.

Some properties of Fourier-Stieltjes transform

Linearity Property

$$FS\{K_1 f_1(t,x) + K_2 f_2(t,x)\} = K_1 FS\{f_1(t,x)\} + K_2 FS\{f_2(t,x)\}$$

Proof

$$FS\{K_1 f_1(t,x) + K_2 f_2(t,x)\} = \int_0^\infty \int_0^\infty [K_1 f_1(t,x) + K_2 f_2(t,x)] e^{-ist}\,(x+y)^{-p}\,dt\,dx$$
$$= K_1 \int_0^\infty \int_0^\infty f_1(t,x)\,e^{-ist}\,(x+y)^{-p}\,dt\,dx$$
$$+ K_2 \int_0^\infty \int_0^\infty f_2(t,x)\,e^{-ist}\,(x+y)^{-p}\,dt\,dx\; FS\{K_1 f_1(t,x) + K_2 f_2(t,x)\} = K_1 FS\{K_1 f_1(t,x)\} + K_2 FS\{K_2 f_2(t,x)\}$$

Scaling Property

$$FS\{f(at,x)\} = \frac{1}{a}\,F(\frac{s}{a},p)$$

Proof

$$FS\{f(at,x)\} = \int_0^\infty \int_0^\infty f(at,x)\,e^{-ist}\,(x+y)^{-p}\,dt\,dx \quad \text{………………….(3.2.1)}$$

Put $at = z \Rightarrow a\,dt = dz$

$$FS\{f(at,x)\} = \int_0^\infty \int_0^\infty f(z,x)\,e^{-is\frac{z}{a}}\,(x+y)^{-p}\,\frac{dz}{a}\,dx$$

Put $\frac{s}{a} = r$

$$FS\{f(at,x)\} = \frac{1}{a}\int_0^\infty \int_0^\infty f(z,x)\,e^{-irz}\,(x+y)^{-p}\,dz\,dx$$
$$FS\{f(at,x)\} = \frac{1}{a}F(r,p)$$
$$FS\{f(at,x)\} = \frac{1}{a}F(\frac{s}{a},p)$$

3.3) I^st Shifting property

$$FS\{e^{-at}f(t,x)\} = F(s-ia,p)$$

Proof

$$FS\{e^{-at}f(t,x)\} = \int_0^\infty \int_0^\infty e^{-at}\,f(t,x)\,e^{-ist}\,(x+y)^{-p}\,dt\,dx$$
$$= \int_0^\infty \int_0^\infty f(t,x)\,e^{-ist-at}\,(x+y)^{-p}\,dt\,dx$$
$$= \int_0^\infty \int_0^\infty f(t,x)\,e^{-i(s-ia)t}\,(x+y)^{-p}\,dt\,dx$$

Put $s - ia = z$

$$FS\{e^{-at}f(t,x)\} = \int_0^\infty \int_0^\infty f(t,x)\,e^{-izt}\,(x+y)^{-p}\,dt\,dx$$
$$= F(z,p)$$
$$FS\{e^{-at}f(t,x)\} = F(s-ia,p)$$

Differential Property

$$\boldsymbol{FS\{f_x(t,x)\} = p\,FS\{f(t,x)\} - k}$$

Proof

$$FS\{f_x(t,x)\} = \int_0^\infty \int_0^\infty f_x(t,x)\,e^{-ist}\,(x+y)^{-p}\,dt\,dx$$
$$= \int_0^\infty e^{-ist}\,dt \int_0^\infty f_x(t,x)\,(x+y)^{-p}\,dx$$

By using integration by parts, we get-

$= \int_0^\infty e^{-ist}\, dt \,\{ (x+y)^{-p} f(t,x))_0^\infty - \int_0^\infty (-p)\,(x+y)^{-(p+1)} f(t,x)dx\}$

$= \int_0^\infty e^{-ist}\, dt \,\{ -y^{-p} f(t,0) + p\int_0^\infty (x+y)^{-(p+1)} f(t,x)dx\}$

$= -\int_0^\infty f(t,0)\, y^{-p} e^{-ist}\, dt \;+ p\int_0^\infty \int_0^\infty e^{-ist}\,(x+y)^{-(p+1)} f(t,x)\, dt\, dx$

Where, $\int_0^\infty f(t,0)\, y^{-p} e^{-ist}\, dt = k$

$FS\{f_x(t,x)\} = -k + p\, FS\{f(t,x)\}$

$FS\{f_x(t,x)\} = p\, FS\{f(t,x)\} - k$

$\boldsymbol{FS\{f_{xx}(t,x)\} = p^2\, FS\{f(t,x)\} - pk}$

Proof

$FS\{f_{xx}(t,x)\} = \int_0^\infty \int_0^\infty f_{xx}(t,x)\, e^{-ist}\,(x+y)^{-p}\, dt\, dx$

$= \int_0^\infty e^{-ist}\, dt \int_0^\infty f_{xx}(t,x)\,(x+y)^{-p}\, dx$

$= \int_0^\infty e^{-ist}\, dt\big[(x+y)^{-p} f_x(t,x))_0^\infty - \int_0^\infty (-p)\,(x+y)^{-(p+1)} f_x(t,x)dx\big]$

$= \int_0^\infty e^{-ist}\, dt\,\{ -y^{-p} f_x(t,0) + p\int_0^\infty (x+y)^{-(p+1)} f_x(t,x)dx\}$

$= -\int_0^\infty f_x(t,0)\, y^{-p} e^{-ist}\, dt \;+ p\int_0^\infty \int_0^\infty e^{-ist}\,(x+y)^{-(p+1)} f_x(t,x)\, dt\, dx$

{Since, by DUIS the value of $\int_0^\infty f_x(t,0)\, y^{-p} e^{-ist}\, dt = 0$ and it is zero for infinite integral or it s ignore}

$= p\int_0^\infty \int_0^\infty e^{-ist}\,(x+y)^{-(p+1)} f_x(t,x)\, dt\, dx$

$= p\,[p\, FS\{f(t,x)\} - k]$

$FS\{f_{xx}(t,x)\} = p^2\, FS\{f(t,x)\} - pk$

Similarly,

$FS\{f_{xxx}(t,x)\} = p^3\, FS\{f(t,x)\} - p^2 k$

$FS\{f_n(t,x)\} = p^n\, FS\{f(t,x)\} - p^{n-1} k$

Differential property for t:

$\boldsymbol{FS\{f_t(t,x)\} = is\, FS\{f(t,x)\} - k}$

Proof

$FS\{f_t(t,x)\} = \int_0^\infty \int_0^\infty f_t(t,x)\, e^{-ist}\,(x+y)^{-p}\, dt\, dx$

$= \int_0^\infty (x+y)^{-p}\, dx \int_0^\infty f_t(t,x)\, e^{-ist}\, dt$

$= \int_0^\infty (x+y)^{-p}\, dx\big[e^{-ist} f(t,x)\}_0^\infty - \int_0^\infty -is\; e^{-ist} f(t,x)dt\big]$

$= \int_0^\infty (x+y)^{-p}\, dx\big[-f(0,x) + is\int_0^\infty e^{-ist} f(t,x)dt\big]$

$= -\int_0^\infty f(0,x)\,(x+y)^{-p}\, dx + is\int_0^\infty \int_0^\infty e^{-ist}(x+y)^{-p}\; f(t,x)dt\; dx$

$FS\{f_t(t,x)\} = is\, FS\{f(t,x)\} - k$, Where, $\int_0^\infty f(0,x)\,(x+y)^{-p}\, dx = k$

$\boldsymbol{FS\{f_{tt}(t,x)\} = (is)^2\, FS\{f(t,x)\} - is\, k}$

Proof

$FS\{f_{tt}(t,x)\} = \int_0^\infty \int_0^\infty f_{tt}(t,x)\, e^{-ist}\,(x+y)^{-p}\, dt\, dx$

$= \int_0^\infty (x+y)^{-p}\, dx \int_0^\infty f_{tt}(t,x)\, e^{-ist}\, dt$

$= \int_0^\infty (x+y)^{-p}\, dx\big[e^{-ist} f_t(t,x))_0^\infty - \int_0^\infty (-is)\, e^{-ist} f_t(t,x)dx\big]$

$= \int_0^\infty (x+y)^{-p}\, dx\,\{ -f_t(0,x) + is\int_0^\infty e^{-ist} f_t(t,x)dx\}$

$= -\int_0^\infty f(0,x)(x+y)^{-p}\, dx \;+ is\int_0^\infty \int_0^\infty e^{-ist}\,(x+y)^{-p} f_t(t,x)\, dt\, dx$

{Since, by DUIS the value of $\int_0^\infty f(0,x)(x+y)^{-p}\, dx = 0$ and it is zero for infinite integral or it s ignore}

$FS\{f_{tt}(t,x)\} = is\int_0^\infty \int_0^\infty e^{-ist}\,(x+y)^{-p} f_t(t,x)\, dt\, dx$

$= is\, FS\{f_t(t,x)\}$

$= is\,(is\, FS\{f(t,x)\} - k)$

$FS\{f_{tt}(t,x)\} = (is)^2\, FS\{f(t,x)\} - is\, k$

Similarly

$FS\{f_{ttt}(t,x)\} = (is)^3\, FS\{f(t,x)\} - (is)^2\, k$
$FS\{f_n(t,x)\} = (is)^n\, FS\{f(t,x)\} - (is)^{n-1}\, k$

Second Shifting Property

If $FS\{f(t,x)\}$ is generalized Fourier-Stieltjes Transform $FS\{f(t-a,x)\} = e^{-isa}\, F(s,p)$

Proof

$FS\{f(t-a,x)\} = \int_0^\infty \int_0^\infty f(t-a,x)\, e^{-ist}\, (x+y)^{-p}\, dt\, dx$
Put, $t-a=z$
$= \int_0^\infty \int_0^\infty f(z,x)\, e^{-is(z+a)}\, (x+y)^{-p}\, dz\, dx$
$= e^{-isa} \int_0^\infty \int_0^\infty f(z,x)\, e^{-isz}\, (x+y)^{-p}\, dz\, dx$
$= e^{-isa} FS\{f(z,x)\}$
$FS\{f(t-a,x)\} = e^{-isa} F(s,p)$

Multiplication by e^{-ibt}

$FS\{e^{-ibt} f(t,x)\} = F(s-b,p)$

Proof

$FS\{e^{-ibt} f(t,x)\} = \int_0^\infty \int_0^\infty f(t-a,x) e^{-ibt}\, e^{-ist}\, (x+y)^{-p}\, dt\, dx$
$= \int_0^\infty \int_0^\infty f(t-a,x) e^{-i(s-b)t}\, (x+y)^{-p}\, dt\, dx$
Put $s-b=z$
$= \int_0^\infty \int_0^\infty f(t-a,x) e^{-izt}\, (x+y)^{-p}\, dt\, dx$
$= FS\{f(t,x)\} = F(z,p)$
$FS\{e^{-ibt} f(t,x)\} = F(s-b,p)$

Multiplication by $(x+y)^m$

$FS\{(x+y)^m\, f(t,x)\} = F(s,p-m)$

Proof

$FS\{(x+y)^m\, f(t,x)\} = \int_0^\infty \int_0^\infty f(t,x)(x+y)^m\, e^{-ist}\, (x+y)^{-p}\, dt\, dx$
$= \int_0^\infty \int_0^\infty f(t,x) e^{-ist}\, (x+y)^{-p+m}\, dt\, dx$
$= \int_0^\infty \int_0^\infty f(t,x) e^{-ist}\, (x+y)^{-(p-m)}\, dt\, dx$
Put $p-m=r$
$FS\{(x+y)^m\, f(t,x)\} = \int_0^\infty \int_0^\infty f(t,x) e^{-ist}\, (x+y)^{-r}\, dt\, dx$
$= F(s,r)$
$FS\{(x+y)^m\, f(t,x)\} = F(s,p-m)$

Conclusion

In the present work, some properties of Distributional Fourier-Stieltjes transform which may be useful in differential and integral equations.

REFERENCES

Debnath L. and D, Batta, 2007. "Integral Transform and their applications", Chapman and Hall/CRC Taylor and Francis Group Boca Raton London, New York.
Gelefand I.M. and G.E. Shilov, "Generalized Function (Vol.-2)", Academic Press, New York.
John H. Schwarz, 2005. "The generalized Stieltjes transform and its inverse", *Journal of Mathematical Physics*, 46, 013501.

Khairnar S.M., R.M. Pise and J.N.Salunke, 2012. "Generalized Finite Mellin Integral Transforms", *Achieves of Applied Sciences Research*, 2012, 4(2), 1127-1134.

Sharma V. D. and A.N. Rangari, 2013. "Operational calculus on Fourier-Finite Mellin transform", *Int. Jr. of Engg. and Innovative Technology*, Vol.3, Issue 4, (2247-3154).

Sharma V.D. and Dolas P.D. , "Generalization of Fourier-Stieltjes transform". *American Jr. of Mathematics and Sciences,* Vol.2, No.1, January 13, ISSN No. 2250-3102.

Sharma V.D. and Dolas P.D. 2013. "Some S-type spaces of Fourier Stieltjes transform", *Int. Jr. of Engineering and Innovative technology* (IJEIT), Vol.3, Issue 3.

Sharma V.D. and Rangari A. N. 2014. "Operational Calculus on Generalized Fourier-Laplace Transform", *International Journal of Scientific and Innovative Mathematical Research* (IJSIMR) Volume 2, Issue 11, PP 862-867.

Sheikh A. and Gudadhe A.S. 2013. "Operation Transform Formulae for Generalized Fractional Hilbert Transform", *Journal of Science and Arts Year* 13, No. 4(25), pp. 339-344.

Shubing Wang, 2007. "Applications of Fourier Transform to Imaging Analysis", May 23.

Zemanian, A.H. 1968. "Generalized Integral Transform", *Inter Science Publishers*, New York, 1968.

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