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RESEARCH ARTICLE

LOCATION OF ZEROS OF POLAR DERIVATIVE OF POLYNOMIAL WITH REAL COEFFICIENTS

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ABSTRACT

In this paper we obtain the size of the disc in which the zeros of polar derivatives of polynomial of degree n with real coefficients with respect to a real α lie.

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INTRODUCTION

To estimate the zeros of a polynomial is a long standing problem. It is an interesting area of research for many engineers as well as mathematicians and many results on the topic are available in the literature.

If $P(z)=\sum_{i=0}^{n} a_i z^i$, be a polynomial of degree n then Polar Derivative of the polynomial P(z) with respect to α, where α can be real or complex number, is defined as

$D_\alpha P(z) = n\,P(z) + (\alpha-z)\,P'(z)$.

It is a polynomial of degree up to n-1. The polynomial $D_\alpha P(z)$ generalizes

the ordinary derivative, in the sense that $\lim_{\alpha\to\infty} D_\alpha P(z)/\alpha = P'(z)$.

The well-known results on Eneström-Kakeya theorem (see [1,2]) in theory of distribution of zeros of polynomials are the following.

Theorem (A_1): Let $P(z) =\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n such that

$0<a_0 \le a_1 \le \dots \le a_n$.

Then all the zeros of P(z) lie in $|z| \le 1$.

A. Joyal, G. Labelle and Q.I. Rahman [3] obtained the following generalization, by considering the coefficients to be real instead of being only positive.

Theorem (A_2): Let $P(z)=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n such that

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$a_0 \leq a_1 \leq \ldots\ldots \leq a_n$.

Then all the zeros of P(z) lie in $|z| \leq |a_n|^{-1}\{a_n - a_0 + |a_0|\}$.

This paper we prove the following results.

Theorem (1): Let $P(z)=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with real coefficients such that for some $\delta \geq 0$

$0<a_0-\delta \leq a_1 \leq \ldots\ldots \leq a_n$

and $(n-i)a_i \leq a_{i+1}$ i= 0, 1, 2, . . .n-1.

Then the polar derivative of P(z) with respect to a positive α has (n-1) roots and they lie in

$|z| \leq (a_{n-1} + \alpha n a_n)^{-1}\{a_{n-1} + \alpha n a_n + 2n\delta\}$.

Corollary (1): Let $P(z)=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with real coefficients such that for

$0<a_0 \leq a_1 \leq \ldots\ldots \leq a_n$

and

$(n-i)a_i \leq a_{i+1}$ i= 0, 1, 2, . . .n-1.

Then the polar derivative of P(z) with respect to a positive α has up to (n-1) roots and they lie in
$|z| \leq 1$.

Remark (1): By taking $\delta=0$ in Theorem (1) we obtain Corollary(1).

Theorem (2): Let $P(z)=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with real coefficients such that for $\delta \geq 0$

$a_0-\delta \leq a_1 \leq \ldots\ldots \leq a_n$

and

$(n-i)a_i \leq a_{i+1}$ i= 0, 1, 2, . . .n-1.

Then the polar derivative of P(z) with respect to any $\alpha \neq -a_{n-1}/na_n$ has (n-1) roots and they lie in

$|z| \leq |a_{n-1} + \alpha n a_n|^{-1}\{a_{n-1} + \alpha n a_n + 2n\delta - na_0 - \alpha a_1 + |na_0 + \alpha a_1|\}$.

Corollary (2): Let $P(z)=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with real coefficients such that for

$a_0 \leq a_1 \leq \ldots\ldots \leq a_n$

and

$(n-i)a_i \leq a_{i+1}$ i= 0, 1, 2, . . .n-1.

Then the polar derivative of P(z) with respect to any $\alpha \neq -a_{n-1}/na_n$ has (n-1) roots and they lie in

$|z| \leq |a_{n-1} + \alpha n a_n|^{-1}\{a_{n-1} + \alpha n a_n - na_0 - \alpha a_1 + |na_0 + \alpha a_1|\}$.

Remark (2): By taking $\delta=0$ in Theorem (2) we obtain Corollary(2).

Theorem (3): Let $P(z)=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with real coefficients such that for

$a_0 - \delta \leq a_1 \leq \ldots\ldots \leq a_{m+1}$ where m= 0, 1, 2……n

and

$(n-i)a_i \leq a_{i+1}$ i= 0, 1, 2, . . .m-1.

Then the polar derivative of P(z) with respect to α such that

$A = -a_{n-1}/ na_n = -2a_{n-2}/(n-1)a_{n-1} = \ldots$

$= -(n-m-1)a_{m+1}/(m+2)a_{m+2} \neq -(n-m)a_m/(m+1)a_{m+1}$

has exactly m roots and they lie in

$|z| \leq |(n-m)a_m + \alpha(m+1)a_{m+1}|^{-1}\{(n-m)a_m + \alpha(m+1)a_{m+1} - na_0 - \alpha na_1 + 2n\delta + |na_0 + \alpha na_1|\}$.

Corollay (3): Let $P(z)=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with real coefficients such that for

$a_0 \leq a_1 \leq \ldots \leq a_{m+1}$ where m= 0, 1, 2……n
and

$(n-i)a_i \leq a_{i+1}$ i= 0, 1, 2, . . .m-1.

Then the polar derivative of P(z) with respect to α such that

$\alpha = -a_{n-1}/ na_n = -2a_{n-2}/(n-1)a_{n-1} = \ldots$

$= -(n-m-1)a_{m+1}/(m+2)a_{m+2} \neq -(n-m)a_m/(m+1)a_{m+1}$

has exactly m roots and they lie in

$|z| \leq |(n-m)a_m + \alpha(m+1)a_{m+1}|^{-1}\{(n-m)a_m + \alpha(m+1)a_{m+1} - na_0 - \alpha na_1 + |na_0 + \alpha na_1|\}$.

Remark (3): By taking δ=0 in Theorem (3) we obtain Corollary(3).

Proof of Theorem 1:

Let $P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$ be a polynomial of degree n.

Then the polar derivative of P(z) is given by $D_\alpha P(z) = n P(z) + (\alpha - z) P'(z)$. Then

$D_\alpha P(z) = [na_0 + \alpha a_1] + [(n-1)a_1 + 2\alpha a_2] z + [(n-2)a_2 + 3\alpha a_3]z^2 + \ldots$

$+ [(n-m+1)a_{m-1} + \alpha m a_m]z^{m-1} + [(n-m)a_m + \alpha(m+1)a_{m+1}]z^m + [(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2}]z^{m+1} + \ldots$

$+ [2a_{n-2} + \alpha(n-1)a_{n-1}]z^{n-2} + [a_{n-1} + \alpha na_n]z^{n-1}$.

Now consider the polynomial $Q(z) = (1-z) D_\alpha P(z)$ so that

$Q(z) = -[a_{n-1} + \alpha na_n]z^n + [a_{n-1} + \alpha na_n - 2a_{n-2} - \alpha(n-1)a_{n-1}]z^{n-1} + \ldots$

$+ [(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2} - (n-m)a_m - \alpha(m+1)a_{m+1}]z^{m+1}$

$+ [(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1} - \alpha m a_m]z^m$

$+ [(n-m+1)a_{m-1} + \alpha m a_m - (n-m+2)a_{m-2} - \alpha(m-1)a_{m-1}]z^{m-1} + \ldots$

$+ [(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2]z^2 + [(n-1)a_1 + 2\alpha a_2 - na_0 - \alpha a_1]z + [na_0 + \alpha a_1]$.

Now if $|z|>1$ then $|z|^{i-n} < 1$ for i = 1, 2, 3, …..n-1

Further

$|Q(z)| \geq |a_{n-1} + \alpha na_n||z|^{n-1} - \{|a_{n-1} + \alpha na_n - 2a_{n-2} - \alpha(n-1)a_{n-1}| \, |z|^{n-1} + \ldots$

$+ |(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2} - (n-m)a_m - \alpha(m+1)a_{m+1}|\, |z|^{m+1}$

$+|(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1} - \alpha m a_m||\, z|^m$

$+|(n-m+1)a_{m-1} + \alpha m a_m - (n-m+2)a_{m-2} - \alpha(m-1)a_{m-1}|\, |z|^{m-1}$

$+|(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2|\, |z|^2 +|(n-1)a_1 + 2\alpha a_2 - na_0 - \alpha a_1|\, |z|$

$+|na_0 + \alpha a_1|\}.$

$\geq |a_{n-1} + \alpha na_n|\, |z|^{n-1}[|z| - | a_{n-1} + \alpha na_n|^{-1}\{|a_{n-1} + \alpha na_n - 2a_{n-2} -\alpha(n-1)a_{n-1}|$

$+ |2a_{n-2} + \alpha(n-1)a_{n-1} -3a_{n-3} -\alpha(n-2)a_{n-2}|\, |z|^{-1} + . . .$

$+|(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2} - (n-m)a_m - \alpha(m+1)a_{m+1}|\, |z|^{-(n-m-2)}$

$+|(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1} - \alpha m a_m||\, z|^{-(n-m-1)}$

$+|(n-m+1)a_{m-1} + \alpha m a_m - (n-m+2)a_{m-2} - \alpha(m-1)a_{m-1}|\, |z|^{-(n-m)} + \ldots$

$+|(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2|\, |z|^{-(n-3)} + |(n-1)a_1 + 2\alpha a_2- na_0 - \alpha a_1|\, |z|^{-(n-2)} +|na_0 + \alpha a_1|\, |z|^{-(n-1)} \}].$

$\geq|a_{n-1} + \alpha na_n|\, |z|^{n-1}[|z| - | a_{n-1} + \alpha na_n|^{-1}\{|a_{n-1} + \alpha na_n - 2a_{n-2} -\alpha(n-1)a_{n-1}|$

$+ |2a_{n-2} + \alpha(n-1)a_{n-1} -3a_{n-3} -\alpha(n-2)a_{n-2}| + . . .$

$+|(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2} - (n-m)a_m - \alpha(m+1)a_{m+1}|$

$+|(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1} - \alpha m a_m |$

$+|(n-m+1)a_{m-1} + \alpha m a_m - (n-m+2)a_{m-2} - \alpha(m-1)a_{m-1}| + \ldots$

$+|(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2| + |(n-1)a_1 + 2\alpha a_2 - na_0+ n\delta - \alpha a_1- n\delta| +|na_0 + \alpha a_1| \}].$

$\geq|a_{n-1} + \alpha na_n|\, |z|^{n-1}[|z| - | a_{n-1} + \alpha na_n|^{-1}\{a_{n-1} + \alpha na_n - 2a_{n-2}$

$-\alpha(n-1)a_{n-1} + 2a_{n-2} + \alpha(n-1)a_{n-1} -3a_{n-3} -\alpha(n-2)a_{n-2} + . . .$

$+(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2} - (n-m)a_m - \alpha(m+1)a_{m+1}$

$+(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1} - \alpha m a_m$

$+(n-m+1)a_{m-1} + \alpha m a_m - (n-m+2)a_{m-2} - \alpha(m-1)a_{m-1} + \ldots$

$+(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2 + (n-1)a_1 + 2\alpha a_2 - na_0+2n\delta - \alpha a_1 +na_0 + \alpha a_1\}].$

$\geq|a_{n-1} + \alpha na_n|\, |z|^{n-1}[|z | -| a_{n-1} + \alpha na_n|^{-1}\{a_{n-1} + \alpha na_n+2n\delta \}].$

>0 if $|z | >| a_{n-1} + \alpha na_n|^{-1}\{a_{n-1} + \alpha na_n+2n\delta \}$

This shows that if $|z | > 1$ then $Q(z) > 0$ if $|z | >| a_{n-1} + \alpha na_n|^{-1}\{a_{n-1} + \alpha na_n+2n\delta \}$.

Hence all the zeros of $Q(z)$ with $|z|>1$ lie in

$|z| \leq| a_{n-1} + \alpha na_n|^{-1}\{a_{n-1} + \alpha na_n+2n\delta \}$

But those zeros of $Q(z)$ whose modulus is less than or equal to 1, already satisfy the above inequality since all the zeros of $D_\alpha P(z)$ are also the zeros of $Q(z)$ as they lie in the circle defined by the above inequality and this completes the proof.

Proof of theorem2:

Let $P(z) = a_0 + a_1z + a_2z^2 + \ldots\ldots + a_nz^n$ be a polynomial of degree n.

Then the polar derivative of P(z) is given by $D_\alpha P(z) = n P(z) + (\alpha - z) P'(z)$. Then

$$D_\alpha P(z) = [na_0 + \alpha a_1] + [(n-1)a_1 + 2\alpha a_2] z + [(n-2)a_2 + 3\alpha a_3]z^2 + \ldots$$

$$+ [(n-m+1)a_{m-1} + \alpha m a_m]z^{m-1} + [(n-m)a_m + \alpha(m+1)a_{m+1}]z^m + [(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2}]z^{m+1} + \ldots\ldots$$

$$+ [2a_{n-2} + \alpha(n-1)a_{n-1}]z^{n-2} + [a_{n-1} + \alpha n a_n]z^{n-1}.$$

Now consider the polynomial $Q(z) = (1-z) D_\alpha P(z)$ so that

$$Q(z) = -[a_{n-1} + \alpha n a_n]z^n + [a_{n-1} + \alpha n a_n - 2a_{n-2} - \alpha(n-1)a_{n-1}]z^{n-1} + \ldots.$$

$$+ [(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2} - (n-m)a_m - \alpha(m+1)a_{m+1}]z^{m+1}$$

$$+ [(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1} - \alpha m a_m]z^m$$

$$+ [(n-m+1)a_{m-1} + \alpha m a_m - (n-m+2)a_{m-2} - \alpha(m-1)a_{m-1}]z^{m-1} + \ldots$$

$$+ [(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2]z^2 + [(n-1)a_1 + 2\alpha a_2 - na_0 - \alpha a_1]z + [na_0 + \alpha a_1].$$

$$Q(z) = -[a_{n-1} + \alpha n a_n][z]z^{n-1} + [a_{n-1} + \alpha n a_n - 2a_{n-2} - \alpha(n-1)a_{n-1}]z^{n-1} + \ldots.$$

$$+ [(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2} - (n-m)a_m - \alpha(m+1)a_{m+1}]z^{m+1}$$

$$+ [(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1} - \alpha m a_m]z^m$$
$$+ [(n-m+1)a_{m-1} + \alpha m a_m - (n-m+2)a_{m-2} - \alpha(m-1)a_{m-1}]z^{m-1} + \ldots$$

$$+ [(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2]z^2 + [(n-1)a_1 + 2\alpha a_2 - na_0 - \alpha a_1]z + [na_0 + \alpha a_1].$$

Now if $|z|>1$ then $|z|^{i-n}< 1$ for $i = 1, 2, 3, \ldots..n-1$

Further

$$|Q(z)| \geq |a_{n-1} + \alpha n a_n||z ||z|^{n-1} - \{|a_{n-1} + \alpha n a_n - 2a_{n-2} - \alpha(n-1)a_{n-1}| |z|^{n-1}$$

$$+ \ldots + |(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2} - (n-m)a_m - \alpha(m+1)a_{m+1}| |z|^{m+1}$$

$$+|(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1} - \alpha m a_m || z|^m$$

$$+|(n-m+1)a_{m-1} + \alpha m a_m - (n-m+2)a_{m-2} - \alpha(m-1)a_{m-1}| |z|^{m-1}$$

$$+|(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2| |z|^2 +|(n-1)a_1 + 2\alpha a_2 - na_0 - \alpha a_1| |z| +|na_0 + \alpha a_1|\}.$$

$$\geq |a_{n-1} + \alpha n a_n| |z|^{n-1}[|z | - | a_{n-1} + \alpha n a_n|^{-1}\{|a_{n-1} + \alpha n a_n - 2a_{n-2} -\alpha(n-1)a_{n-1}|$$

$$+ |2a_{n-2} + \alpha(n-1)a_{n-1} -3a_{n-3} -\alpha(n-2)a_{n-2}| |z|^{-1} + \ldots$$

$$+|(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2} - (n-m)a_m - \alpha(m+1)a_{m+1}| |z|^{-(n-m-2)}$$

$$+|(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1} - \alpha m a_m || z|^{-(n-m-1)}$$

$$+|(n-m+1)a_{m-1} + \alpha m a_m - (n-m+2)a_{m-2} - \alpha(m-1)a_{m-1}| |z|^{-(n-m)} + \ldots$$

$$+|(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2| |z|^{-(n-3)} + |(n-1)a_1 + 2\alpha a_2- na_0 - \alpha a_1| |z|^{-(n-2)} +|na_0 + \alpha a_1| |z|^{-(n-1)} \}].$$

$$\geq|a_{n-1} + \alpha n a_n| |z|^{n-1}[|z | - | a_{n-1} + \alpha n a_n|^{-1}\{|a_{n-1} + \alpha n a_n - 2a_{n-2} -\alpha(n-1)a_{n-1}|$$

$$+ |2a_{n-2} + \alpha(n-1)a_{n-1} -3a_{n-3} -\alpha(n-2)a_{n-2}| + \ldots$$

$$+|(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2} - (n-m)a_m - \alpha(m+1)a_{m+1}|$$

$+|(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1} - \alpha m a_m|$

$+|(n-m+1)a_{m-1} + \alpha m a_m - (n-m+2)a_{m-2} - \alpha(m-1)a_{m-1}| + \ldots$

$+|(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2| + |(n-1)a_1 + 2\alpha a_2 - na_0 + n\delta - \alpha a_1 - n\delta| + |na_0 + \alpha a_1| \}].$

$\geq |a_{n-1} + \alpha na_n|\, |z|^{n-1}[|z| - |a_{n-1} + \alpha na_n|^{-1}\{a_{n-1} + \alpha na_n - 2a_{n-2} - \alpha(n-1)a_{n-1} + 2a_{n-2} + \alpha(n-1)a_{n-1} - 3a_{n-3} - \alpha(n-2)a_{n-2}$

$+\ldots+(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2} - (n-m)a_m - \alpha(m+1)a_{m+1}$

$+(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1} - \alpha m a_m$

$+(n-m+1)a_{m-1} + \alpha m a_m - (n-m+2)a_{m-2} - \alpha(m-1)a_{m-1} + \ldots$

$+(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2 + (n-1)a_1 + 2\alpha a_2 - na_0 - \alpha a_1 + 2n\delta + |na_0 + \alpha a_1|\}].$

$\geq |a_{n-1} + \alpha na_n|\, |z|^{n-1}[|z| - |a_{n-1} + \alpha na_n|^{-1}\{a_{n-1} + \alpha na_n - na_0 - \alpha a_1 + 2n\delta + |na_0 + \alpha a_1|\}].$

>0 if $|z| > |a_{n-1} + \alpha na_n|^{-1}\{a_{n-1} + \alpha na_n - na_0 - \alpha a_1 + 2n\delta + |na_0 + \alpha a_1|\}$

This shows that if

$|z| > |a_{n-1} + \alpha na_n|^{-1}\{a_{n-1} + \alpha na_n) - na_0 - \alpha a_1 + 2n\delta + |na_0 + \alpha a_1|\}$ then $Q(z) > 0$.

Hence all the zeros of Q(z) with $|z|>1$ lie in

$|z| \leq |a_{n-1} + \alpha na_n|^{-1}\{a_{n-1} + \alpha na_n - na_0 - \alpha a_1 + 2n\delta + |na_0 + \alpha a_1|\}$.

But those zeros of Q(z) whose modulus is less than or equal to 1, already satisfy the above inequality since all the zeros of $D_\alpha P(z)$ are also the zeros of Q(z) as they lie in the circle defined by the above inequality and this completes the proof.

Proof of theorem3:

Let $P(z) = a_0 + a_1z + a_2z^2 + \ldots\ldots + a_nz^n$ be a polynomial of degree n.

Then the polar derivative of P(z) is given by $D_\alpha P(z) = n P(z) + (\alpha-z) P'(z)$. Then

$D_\alpha P(z) = [na_0 + \alpha a_1] + [(n-1)a_1 + 2\alpha a_2] z + [(n-2)a_2 + 3\alpha a_3]z^2 + \ldots$

$+ [(n-m+1)a_{m-1} + \alpha m a_m]z^{m-1} + [(n-m)a_m + \alpha(m+1)a_{m+1}]z^m + [(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2}]z^{m+1} + \ldots\ldots + [2a_{n-2} + \alpha(n-1)a_{n-1}]z^{n-2} + [a_{n-1} + \alpha na_n]z^{n-1}$.

As $\alpha = -a_{n-1}/na_n = -2a_{n-2}/(n-1)a_{n-1} = \ldots = -(n-m-1)a_{m+1}/(m+2)a_{m+2} \neq -(n-m)a_m/(m+1)a_{m+1}$

$D_\alpha P(z) = [(n-m)a_m + \alpha(m+1)a_{m+1}]z^m + [(n-m+1)a_{m-1} + \alpha m a_m]z^{m-1} + \ldots + [(n-1)a_1 + 2\alpha a_2] z + [na_0 + \alpha a_1]$.

Now consider the polynomial $Q(z) = (1-z) D_\alpha P(z)$ so that

$Q(z) = -[(n-m)a_m + \alpha(m+1)a_{m+1}]z^{m+1} + [(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1}$

$- \alpha m a_m]z^m + [(n-m+1)a_{m-1} + \alpha m a_m - (n-m+2)a_{m-2} - \alpha(m-1)a_{m-1}]z^{m-1} + \ldots$

$+ [(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2]z^2 + [(n-1)a_1 + 2\alpha a_2 - na_0 - \alpha a_1]z + [na_0 + \alpha a_1]$.

Now if $|z|>1$ then $|z|^{i-m} < 1$ for i = 1, 2, 3, …..m-1

Further,

$|Q(z)| \geq |(n-m)a_m + \alpha(m+1)a_{m+1}||z|^{m+1} - \{|(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1} - \alpha m a_m|\, |z|^m$

$+ |(n-m+1)a_{m-1} + \alpha m a_m - (n-m+2)a_{m-2} - \alpha(m-1)a_{m-1}|\, |z|^{m-1} + \ldots$

$+|(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2| |z|^2 + |(n-1)a_1 + 2\alpha a_2 -na_0 - \alpha a_1 | |z| + |na_0 + \alpha a_1|.$

$\geq |(n-m)a_m + \alpha(m+1)a_{m+1}||z|^m \; [|z| -|(n-m)a_m + \alpha(m+1)a_{m+1}|^{-1}$

$\{|(n-m)a_m + \alpha(m+1)a_{m+1}- (n-m+1)a_{m-1} - \alpha ma_m |+\ldots$

$+|(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2| |z|^{-(m-2)}$

$+|(n-1)a_1 + 2\alpha a_2 -na_0 - \alpha a_1 ||z |^{-(m-1)}+|na_0 + \alpha a_1||z|^{-m}\}].$

$\geq |(n-m)a_m + \alpha(m+1)a_{m+1}||z|^m \; [|z| -|(n-m)a_m + \alpha(m+1)a_{m+1}|^{-1}$

$\{|(n-m)a_m + \alpha(m+1)a_{m+1}- (n-m+1)a_{m-1} - \alpha ma_m |+\ldots$

$+|(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2|$

$+|(n-1)a_1 + 2\alpha a_2 - na_+ n\delta - \alpha a_1-n\delta |+|na_0 + \alpha a_1|\}].$

$\geq |(n-m)a_m + \alpha(m+1)a_{m+1}||z|^m \; [|z|$

$-|(n-m)a_m + \alpha(m+1)a_{m+1}|^{-1} \{(n-m)a_m + \alpha(m+1)a_{m+1}$

$-(n-m+1)a_{m-1} - \alpha ma_m +\ldots +(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2$

$+(n-1)a_1 + 2\alpha a_2 -na_0 - \alpha a_1+2n\delta+|na_0 + \alpha a_1|\}].$

$\geq |(n-m)a_m + \alpha(m+1)a_{m+1}||z|^m \; [|z|$

$-|(n-m)a_m + \alpha(m+1)a_{m+1}|^{-1} \{(n-m)a_m + \alpha(m+1)a_{m+1} -na_0 - \alpha a_1+2n\delta+|na_0 + \alpha a_1|\}].$

>0 if $|z| > |(n-m)a_m + \alpha(m+1)a_{m+1}|^{-1} \{ (n-m)a_m + \alpha(m+1)a_{m+1}- na_0-\alpha a_1 +2n\delta +|na_0 + \alpha a_1|\}.$

This shows that if

$|z| > |(n-m)a_m + \alpha(m+1)a_{m+1}|^{-1} \{ (n-m)a_m +\alpha(m+1)a_{m+1} -na_0 - \alpha a_1+2n\delta +|na_0 + \alpha a_1|\}.$

then $Q(z) > 0$.

Hence all the zeros of $Q(z)$ with $|z|>1$ lie in

$|z| >|(n-m)a_m + \alpha(m+1)a_{m+1}|^{-1} \{ (n-m)a_m + \alpha(m+1)a_{m+1} -na_0 -\alpha a_1+2n\delta +|na_0 + \alpha a_1|\}.$

But those zeros of $Q(z)$ whose modulus is less than or equal to 1, already satisfy the above inequality since all the zeros of $D_\alpha P(z)$ are also the zeros of $Q(z)$ as they lie in the circle defined by the above inequality and this completes the proof.

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