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RESEARCH ARTICLE

A COMMON COUPLED FIXED POINT RESULT IN COMPLEX VALUED METRIC SPACE FOR TWO MAPPINGS

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ABSTRACT

In 2006 Bhaskar and Laxmikantham introduced the concept of coupled fixed point for a partially ordered set. In this paper a common coupled fixed point result for two mappings are presented.

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INTRODUCTION

One of the main pillar in the study of fixed point theory is Banach Contraction priciple which was done by Banach in 1922. Fixed point theory is very usefull in various branches of mathematics and science. In 2011 Akbar Azam *et al.,* 2011 introduced the concept of complex valued metric space. The concept of coupled fixed point was first introduced by Bhaskar and Laxikantham in 2006. Recently some researchers prove some coupled fixed point theorems in complex valued metric space in (Kang *et al.,* 2013; Savitri, 2015). The main purpous of this paper is to obtain a common coupled fixed point result in complex valued metric space.

Let $\mathbb{C}$ be the set of all complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order relation $\precsim$ on $\mathbb{C}$ as follows:

$z_1 \precsim z_2$ if and only if $Re(z_1) \leq Re(z_2)$ and $Im(z_1) \leq Im(z_2)$.

Thus $z_1 \precsim z_2$ if one of the followings holds:

- $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$,
- $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$,
- $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$ and
- $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$

We write $z_1 \precnsim z_2$ if $z_1 \precsim z_2$ and $z_1 \neq z_2$ i.e., one of (2), (3) and (4) is satisfied and we will write $z_1 \prec z_2$ if only (4) is satisfied.

Remark 1: We can easily check the followings:

- $a, b \in \mathbb{R}, a \leq b \Rightarrow az \precsim bz \ \forall \, z \in \mathbb{C}$.
- $0 \precsim z_1 \precnsim z_2 \Rightarrow |z_1| < |z_2|$.
- $z_1 \precsim z_2$ and $z_2 \prec z_3 \Rightarrow z_1 \prec z_3$.

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Azam *et al.*, 2011 defined the complex valued metric space in the following way:

Definition 1 ([1]): Let X be a nonempty set. Suppose that the mapping $d : X \times X \to \mathbb{C}$ satisfies the following conditions:

- (C1) $0 \precsim d(x,y)$, for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$;
- (C2) $d(x,y) = d(y,x)$ for all $x, y \in X$;
- (C3) $d(x,y) \precsim d(x,z) + d(z,y)$ for all $x, y, z \in X$.

Then d is called a complex valued metric on X and (X,d) is called a complex valued metric space.

Example 1.1: Let $= \mathbb{R}$. Define the mapping $d : X \times X \to \mathbb{C}$ by

$d(x,y) = \log z\,|x - y|,\ \forall\ x, y \in \mathbb{R}$,

Where z is a fixed complex number, such that $0 < \arg(z) < \frac{\pi}{2}$ and $|z| > 1$ [Here logarithm takes only the principle value].

Then clearly we can show that (X,d) is a complex valued metric space.

Definition 2 ([1]): Let (X,d) be a complex valued metric space. Then

(i)A point $x \in X$ is called an interior point of a set $A \subseteq X$ if there exists $0 \prec r \in \mathbb{C}$ such that

$$B(x,r) = \{y \in X : d(x,y) \prec r\} \subseteq A.$$

A subset $A \subseteq X$ is called open if each element of A is an interior point of A.

(ii)A point $x \in X$ is called a limit point of $A \subseteq X$ if for every $0 \prec r \in \mathbb{C}$,

$$B(x,r) \cap (A - \{x\}) \neq \phi.$$

A subset $A \subseteq X$ is called closed if each element of $X - A$ is not a limit point of A.

(iii)The family

$$F = \{B(x,r) : x \in X, 0 \prec r\}$$

is a sub-basis for a Hausdorff topology τ on X.

Definition 3 ([1]): Let (X,d) be a complex valued metric space. Then

(i)A sequence $\{x_n\}$ in X is said to converge to $x \in X$ if for every $0 \prec r \in \mathbb{C}$ there exists $N \in \mathbb{N}$ such that $d(x_n, x) \prec r,\ \forall\ n > N$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(ii)If for every $0 \prec r \in \mathbb{C}$ there exists $N \in \mathbb{N}$ such that $d(x_n, x_{n+m}) \prec r$ for all $n > N, m \in \mathbb{N}$, then $\{x_n\}$ is called a Cauchy sequence in (X,d).

(iii)If every Cauchy sequence in X is convergent in X then (X,d) is called a complete complex valued metric space.

Definition 4[2]: Let (X,d) be a metric space. An element $(x,y) \in X \times X$ is called a coupled fixed point of the mapping $T : X \times X \to X$ if

$$x = T(x,y) \text{ and } y = T(y,x).$$

Definition 5[4]: Let (X,d) be a complex valued metric space. An element $(x,y) \in X \times X$ is called a coupled fixed point of the mapping $T : X \times X \to X$ if

$$x = T(x,y) \text{ and } y = T(y,x).$$

Example 4.1: Let $X = \mathbb{R}$ with complex valued metric d defined as, $d(x,y) = i|x - y|$. Let $T : X \times X \to X$ defined as $T(x,y) = x^2y^3$. Then $(0,0)$ and $(1,1)$ are two coupled fixed points of T.

Lemma 1 ([1]): Let (X,d) be a complex valued metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to $x \in X$ if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.

Lemma 2 ([1]): Let (X,d) be a complex valued metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$ where $m \in \mathbb{N}$.

Lemma 3([3]): Let (X,d) be a complex valued metric space and $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} x_n = x$. Then for any $a \in X$, $\lim_{n\to\infty} d(x_n, a) = d(x,a)$.

MAIN RESULT

In this section we present the main result.

Theorem: Let (X,d) be a complex valued metric space. Let $S, T : X \times X \to X$, such that

$$d\big(S(x,y),T(u,v)\big) \precsim a\frac{d(x,u)+d(y,v)}{2}+\frac{bd\big(x,S(x,y)\big)d\big(u,T(u,v)\big)+cd\big(u,S(x,y)\big)d\big(x,T(u,v)\big)}{1+d(x,u)+d(y,v)}, \forall\, x,y,u,v \in X,$$

Where a, b, c are non-negative reals satisfying $a+b<1$ and $a+c<1$. Then the mappings S and T have a unique common coupled fixed point in $X \times X$.

Proof: Let $x_0, y_0 \in X$. Let us define two sequences $\{x_n\}$ and $\{y_n\}$ in X as follows.

$x_{2k+1} = S(x_{2k}, y_{2k})$, $x_{2k+2} = T(x_{2k+1}, y_{2k+1})$ and
$y_{2k+1} = S(y_{2k}, x_{2k})$, $y_{2k+2} = T(y_{2k+1}, x_{2k+1})$

Now,

$$d(x_{2k+1},x_{2k+2}) = d(S(x_{2k},y_{2k}),T(x_{2k+1},y_{2k+1}))$$

$$\precsim a\frac{d(x_{2k},x_{2k+1})+d(y_{2k},y_{2k+1})}{2}+ \frac{bd\big(x_{2k},S(x_{2k},y_{2k})\big)d\big(x_{2k+1},T(x_{2k+1},y_{2k+1})\big)+cd\big(x_{2k+1},S(x_{2k},y_{2k})\big)d(x_{2k},T(x_{2k+1},y_{2k+1}))}{1+d(x_{2k},x_{2k+1})+d(y_{2k},y_{2k+1})}$$

$$= a\frac{d(x_{2k},x_{2k+1})+d(y_{2k},y_{2k+1})}{2}+\frac{bd(x_{2k},x_{2k+1})d(x_{2k+1},x_{2k+2})+cd(x_{2k+1},x_{2k+1})d(x_{2k},x_{2k+2})}{1+d(x_{2k},x_{2k+1})+d(y_{2k},y_{2k+1})}$$

$$\precsim a\frac{d(x_{2k},x_{2k+1})+d(y_{2k},y_{2k+1})}{2}+bd(x_{2k+1},x_{2k+2})$$

Thus,

$$d(x_{2k+1},x_{2k+2}) \precsim \frac{a[d(x_{2k},x_{2k+1})+d(y_{2k},y_{2k+1})]}{2(1-b)} \quad \ldots\ldots \quad (1)$$

Similarly, we can show that,

$$d(y_{2k+1},y_{2k+2}) \precsim \frac{a[d(x_{2k},x_{2k+1})+d(y_{2k},y_{2k+1})]}{2(1-b)} \quad \ldots\ldots \quad (2)$$

Adding (1) and (2) we get,

$$d(x_{2k+1},x_{2k+2}) + d(y_{2k+1},y_{2k+2}) \precsim \frac{a[d(x_{2k},x_{2k+1})+d(y_{2k},y_{2k+1})]}{(1-b)}$$

Now, let $h=\frac{a}{1-b}$. Then $0 \le h < 1$ as $0 \le a+b < 1$.

Then,

$$d(x_{2k+1},x_{2k+2}) + d(y_{2k+1},y_{2k+2}) \precsim h[d(x_{2k},x_{2k+1})+d(y_{2k},y_{2k+1})].$$

Similarly,

$$d(x_{2k+2},x_{2k+3}) = d(S(x_{2k+2},y_{2k+2}),T(x_{2k+1},y_{2k+1}))$$

$$\precsim a\frac{d(x_{2k+2},x_{2k+1})+d(y_{2k+2},y_{2k+1})}{2}+ \frac{bd\big(x_{2k+2},S(x_{2k+2},y_{2k+2})\big)d\big(x_{2k+1},T(x_{2k+1},y_{2k+1})\big)+cd\big(x_{2k+1},S(x_{2k+2},y_{2k+2})\big)d(x_{2k+2},T(x_{2k+1},y_{2k+1}))}{1+d(x_{2k+2},x_{2k+1})+d(y_{2k+2},y_{2k+1})}$$

$$= a\frac{d(x_{2k+2},x_{2k+1})+d(y_{2k+2},y_{2k+1})}{2}+\frac{bd(x_{2k+2},x_{2k+3})d(x_{2k+1},x_{2k+2})+cd(x_{2k+1},x_{2k+3})d(x_{2k+2},x_{2k+2})}{1+d(x_{2k+2},x_{2k+1})+d(y_{2k+2},y_{2k+1})}$$

$$\precsim a\frac{d(x_{2k+2},x_{2k+1})+d(y_{2k+2},y_{2k+1})}{2}+bd(x_{2k+2},x_{2k+3})$$

Thus,

$$d(x_{2k+2},x_{2k+3}) \precsim \frac{a[d(x_{2k+2},x_{2k+1})+d(y_{2k+2},y_{2k+1})]}{2(1-b)} \quad \ldots\ldots \quad (3)$$

Similarly, we can show that,

$$d(y_{2k+2}, y_{2k+3}) \precsim \frac{a[d(x_{2k+2}, x_{2k+1}) + d(y_{2k+2}, y_{2k+1})]}{2(1-b)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

Adding (3) and (4) we get,

$$d(x_{2k+2}, x_{2k+3}) + d(y_{2k+2}, y_{2k+3}) \precsim \frac{a[d(x_{2k+2}, x_{2k+1}) + d(y_{2k+2}, y_{2k+1})]}{(1-b)}$$

Thus,

$$d(x_{2k+2}, x_{2k+3}) + d(y_{2k+2}, y_{2k+3}) \precsim h[d(x_{2k+2}, x_{2k+1}) + d(y_{2k+2}, y_{2k+1})]$$

Now, for $n \in \mathbb{N}$, we have,

$$\begin{aligned} d(x_{n+2}, x_{n+1}) + d(y_{n+2}, y_{n+1}) &\precsim h[d(x_{n+1}, x_n) + d(y_{n+1}, y_n)] \\ &\precsim h^2[d(x_n, x_{n-1}) + d(y_n, y_{n-1})] \\ &\precsim \cdots \ldots\ldots\ldots\ldots\ldots\ldots \\ &\precsim h^{n+1}[d(x_1, x_0) + d(y_1, y_0)] \end{aligned}$$

Thus, for $m > n$,

$$\begin{aligned} d(x_m, x_n) + d(y_m, y_n) &\precsim [d(x_n, x_{n+1}) + d(y_n, y_{n+1})] + [d(x_{n+1}, x_m) + d(y_{n+1}, y_m)] \\ &\precsim [d(x_n, x_{n+1}) + d(y_n, y_{n+1})] + [d(x_{n+1}, x_{n+2}) + d(y_{n+1}, y_{n+2})] + \\ &\qquad [d(x_{n+2}, x_m) + d(y_{n+2}, y_m)] \\ &\precsim \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\precsim [d(x_n, x_{n+1}) + d(y_n, y_{n+1})] + [d(x_{n+1}, x_{n+2}) + d(y_{n+1}, y_{n+2})] + \cdots \ldots\ldots \\ &\qquad + [d(x_{m-1}, x_m) + d(y_{m-1}, y_m)] \\ &\precsim [h^n + h^{n+1} + \cdots \ldots\ldots\ldots + h^{m-1}][d(x_1, x_0) + d(y_1, y_0)] \\ &\precsim \frac{h^n}{1-h}[d(x_1, x_0) + d(y_1, y_0)] \\ &\longrightarrow 0 \text{ as } n \longrightarrow \infty. \end{aligned}$$

Thus, $d(x_m, x_n) \longrightarrow 0$ and $d(y_m, y_n) \longrightarrow 0$ as $m, n \longrightarrow \infty$.

Therefore $\{x_n\}, \{y_n\}$ are Cauchy sequences in X.
Since X is complete, then there exist $x, y \in X$, such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$.

Now,

$$\begin{aligned} d(S(x,y), x) &\precsim d(S(x,y), x_{2k+2}) + d(x_{2k+2}, x) \\ &= d\big(S(x,y), T(x_{2k+1}, y_{2k+1})\big) + d(x_{2k+2}, x) \\ &\precsim a\frac{d(x, x_{2k+1}) + d(y, y_{2k+1})}{2} + \frac{bd\big(x, S(x,y)\big)d\big(x_{2k+1}, T(x_{2k+1}, y_{2k+1})\big) + cd\big(x_{2k+1}, S(x,y)\big)d\big(x, T(x_{2k+1}, y_{2k+1})\big)}{1 + d(x, x_{2k+1}) + d(y, y_{2k+1})} + d(x_{2k+2}, x) \\ &= a\frac{d(x, x_{2k+1}) + d(y, y_{2k+1})}{2} + \frac{bd\big(x, S(x,y)\big)d(x_{2k+1}, x_{2k+2}) + cd\big(x_{2k+1}, S(x,y)\big)d(x, x_{2k+2})}{1 + d(x, x_{2k+1}) + d(y, y_{2k+1})} + d(x_{2k+2}, x) \end{aligned}$$

Letting $k \longrightarrow \infty$, we get,

$d(S(x,y), x) \precsim 0.$

Thus $d(S(x,y), x) = 0$ and hence $S(x,y) = x$.

Similarly, we can show that, $S(y,x) = y$.

Again,
$d\big(x, T(x,y)\big) = d\big(S(x,y), T(x,y)\big)$

$$\precsim a\frac{d(x,x)+d(y,y)}{2}+\frac{bd\big(x,S(x,y)\big)d\big(x,T(x,y)\big)+cd\big(x,S(x,y)\big)d\big(x,T(x,y)\big)}{1+d(x,x)+d(y,y)}$$

$$= bd(x,x)d\big(x,T(x,y)\big)+cd(x,x)d\big(x,T(x,y)\big)$$

$$= 0$$

Thus $d\big(x,T(x,y)\big)\precsim 0$. Therefore $d\big(x,T(x,y)\big)=0$ and hence $T(x,y)=x$.

Similarly, we can show that, $T(y,x)=y$.

Thus (x,y) is a common coupled fixed point of S and T.

Now we show that (x,y) is the unique common coupled fixed point of S and T.

If possible, let (x^*,y^*) be another common coupled fixed point of S and T.

Then,

$S(x^*,y^*)=T(x^*,y^*)=x^*$ and $S(y^*,x^*)=T(y^*,x^*)=y^*$.

Now,

$$d(x,x^*)=d\big(S(x,y),T(x^*,y^*)\big)$$

$$\precsim a\frac{d(x,x^*)+d(y,y^*)}{2}+\frac{bd\big(x,S(x,y)\big)d\big(x^*,T(x^*,y^*)\big)+cd\big(x^*,S(x,y)\big)d\big(x,T(x^*,y^*)\big)}{1+d(x,x^*)+d(y,y^*)}$$

$$\precsim a\frac{d(x,x^*)+d(y,y^*)}{2}+\frac{bd(x,x)d(x^*,x^*)+cd(x^*,x)d(x,x^*)}{1+d(x,x^*)+d(y,y^*)}$$

$$\precsim \frac{a}{2}[d(x,x^*)+d(y,y^*)]+cd(x,x^*) \qquad \dots\dots\dots (5)$$

Similarly, we can show that,

$$d(y,y^*)\precsim \frac{a}{2}[d(x,x^*)+d(y,y^*)]+cd(y,y^*) \qquad \dots\dots\dots (6)$$

Adding (5) and (6) , we get,

$d(x,x^*)+d(y,y^*)\precsim (a+c)[d(x,x^*)+d(y,y^*)]$

Thus, $(1-a-c)[d(x,x^*)+d(y,y^*)]\precsim 0$

Since $0\le a+c<1$, then $[d(x,x^*)+d(y,y^*)]=0$.
Thus, $d(x,x^*)=0=d(y,y^*)$.

Hence $x=x^*$ and $y=y^*$. i.e. $(x,y)=(x^*,y^*)$.

Thus (x,y) is the unique common fixed point of S and T.

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