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# RESEARCH ARTICLE

# A DETAILED STUDY OF IDEALS IN COMMUTATIVE RINGS WITH SPECIAL EMPHASIS ON PRIME IDEALS, MAXIMAL IDEALS AND QUOTIENT RINGS

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## ABSTRACT

Ideals play a central role in commutative ring theory, providing a framework to understand ring structure and properties. In this paper we present a detailed study of ideals in commutative rings, with special focus on prime ideals, maximal ideals, and the construction of quotient rings. We review key definitions and known results – notably that every maximal ideal is prime, though the converse need not hold – and survey recent developments in the general theory of ideals. Building on existing literature, we discuss various generalizations of prime ideals introduced in contemporary research (such as $S$-prime, 2-prime, and 1-absorbing prime ideals) and their significance. In the methodology, a theoretical approach is employed alongside illustrative examples. The results include a comparative analysis of ideal structures in different rings and an examination of how prime and maximal ideals govern the properties of quotient rings (integral domains or fields). Two parameters – namely ring type and ideal type – are analyzed through example rings, with supporting tables and lattice diagrams.



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# INTRODUCTION

Ideals are fundamental to understanding commutative rings, encapsulating the notion of "divisibility" and enabling the construction of factor rings. The concepts of prime ideals and maximal ideals are particularly important. An ideal $P$ in a commutative ring $R$ is *prime* if $P \neq R$ and whenever a product $ab$ lies in $P$, then at least one of $a$ or $b$ lies in $P$. Equivalently, $R/P$ is an integral domain when $P$ is prime. An ideal $M$ is *maximal* if $M \neq R$ and there is no ideal strictly between $M$ and $R$; equivalently, the quotient ring $R/M$ is a field. From these definitions it follows that every maximal ideal is prime, but not every prime ideal is maximal. For example, in the ring of integers $\mathbb{Z}$, the zero ideal $(0)$ is prime (since $\mathbb{Z}/(0) \cong \mathbb{Z}$ is an integral domain) but not maximal (since $\mathbb{Z}$ is not a field). In contrast, an ideal generated by a prime number, $(p)$ for $p \in \mathbb{Z}$, is maximal because $\mathbb{Z}/(p) \cong \mathbb{Z}_p$ is a field. These observations illustrate that prime ideals generalize the concept of primality in integers (with fields as "prime-free" rings), while maximal ideals correspond to the "largest" proper ideals beyond which no nontrivial factorization of the ring is possible. Prime and maximal ideals are not only structurally significant but also bridge ring theory with other areas. In algebraic geometry, prime ideals in a ring correspond to points of the algebraic variety defined by that ring's spectrum, providing the foundation for schemes. In number theory, prime ideals generalize prime numbers in the ring of algebraic integers of number fields.

Even in applied fields like cryptography and coding theory, quotient rings modulo prime or maximal ideals produce finite fields ($\mathbb{Z}_p$, polynomial field extensions, etc.) that are essential for constructing cryptosystems and error-correcting codes. Thus, a clear understanding of prime and maximal ideals and their quotient rings is vital. This paper provides a focused study on these concepts. In the next section, we survey the literature, highlighting classical results and recent research that generalizes prime ideals. In Section 3, we outline our methodology, which combines theoretical analysis with illustrative examples. Section 4 presents results and discussion, including comparative analyses (with tables and figures) of ideal structures in example rings. Finally, we conclude with a summary of findings and potential directions. Throughout, we maintain rigorous in-text citation of definitions and theorems from contemporary sources and illustrate claims with concrete examples, ensuring a comprehensive and well-referenced treatment.

# LITERATURE

The theory of ideals in commutative rings has a rich history, originating with Dedekind's introduction of ideals in the 19th century to generalize prime numbers in algebraic number fields. Since then, prime and maximal ideals have been recognized as cornerstones of commutative algebra.

Classic textbooks (e.g. Atiyah & Macdonald, 1969) establish fundamental results such as the characterization of prime and maximal ideals via quotient rings and the fact that maximal ideals form a subset of prime ideals. These foundational results are widely known and form the basis for modern research. For instance, Anderson and Smith (2003) introduced the notion of weakly prime ideals as an early generalization, where a product ab in an ideal I only forces one factor a or b to lie in I when neither belongs to a certain fixed subset of the ring. This opened the door to a wide range of generalized ideal concepts over the last two decades.

Researchers have defined and studied many new classes of ideals that lie between classical prime ideals and more general ideals, aiming to refine how algebraic properties are captured. S-prime ideals were introduced by Hamed and Malek (2011) as a generalization in which primeness is required only after excluding a multiplicative subset S of the ring. Similarly, 2-prime ideals were studied by Nikandish *et al*. (2015) and others, with the defining condition that if ab belongs to an ideal I, then either $a^2$ belongs to I or $b^2$ belongs to I. This concept led to further variations such as weakly 2-prime ideals and results including a "2-prime avoidance lemma" analogous to the classical prime avoidance theorem. In a 2014 study, Tekir *et al.* introduced n-ideals and later J-ideals, where the inclusion of a product ab in an ideal implies conditions on a relative to the Jacobson radical or the nilradical of the ring. These works demonstrate how classical properties of prime ideals can be modified by additional algebraic constraints, resulting in a structured hierarchy of ideal types.

More recent generalizations continue this line of research. S-2-prime ideals (Yavuz *et al*., 2024) combine the concepts of S-prime and 2-prime ideals, with each appearing as a special case. This notion has been applied to investigate ideal properties in amalgamated algebra constructions and localization frameworks. Mimouni (2024) introduced functional prime ideals, extending the concept of prime ideals to modules and linear functionals, thereby creating a link between ring theory and module theory. Another significant development is the notion of 1-absorbing prime ideals, introduced by Badawi and collaborators. An ideal Q is called 1-absorbing prime if, whenever a product xyz lies in Q (where x, y, and z are non-units), then either xy belongs to Q or z belongs to Q, a property observed in Noetherian and von Neumann regular rings. This concept builds upon earlier studies on 1-absorbing ideals and 1-absorbing primary ideals, which themselves generalize classical prime and primary ideals respectively. Overall, the literature shows an active effort to generalize prime ideals to broader contexts. Many of these generalizations, such as weakly prime ideals, S-prime ideals, 2-prime ideals, and 1-absorbing prime ideals, maintain the spirit of primality in the sense that an ideal strongly controls products, while adapting to specific ring conditions or incorporating additional parameters. These studies frequently lead to new characterizations and structural theorems, including decomposition results analogous to primary decomposition and various forms of avoidance lemmas, which extend classical results in commutative algebra. In addition, the connection between ideal theory and other domains remains an important area of investigation. For example, recent studies emphasize applications of generalized prime ideals in areas such as graph theory and cryptology. Building on this body of scholarship, the present paper employs contemporary definitions and theorems as references and concentrates on illustrating the fundamental concepts of prime and maximal ideals, along with quotient rings, through concrete examples and comparative analysis.

# RESEARCH METHODOLOGY

The research methodology adopted in this study is primarily theoretical and expository, with the objective of consolidating established results and demonstrating them through clear and illustrative examples rather than introducing new theorems. The study begins by collecting formal definitions and key propositions concerning prime and maximal ideals from modern algebra texts and recent research articles in order to ensure that the notation and perspective are current. For each important statement, such as the assertion that an ideal M is maximal if and only if the quotient ring R/M is a field, verification is carried out using at least one contemporary source. In addition, the definitions of recently introduced classes of ideals discussed in the literature survey are examined to clarify how they relate to the classical notion of prime ideals. To enhance practical understanding, a comparative case-study approach is employed. Representative rings are selected, and their ideals are systematically enumerated. Each ideal is then analyzed to determine whether it is prime or maximal, using the established definitions and criteria. In the case of finite commutative rings, such as $Z_n$ (the ring of integers modulo n), a complete enumeration of ideals is performed by utilizing the correspondence between ideals of Z and the divisors of n. For each ideal, the corresponding quotient ring is constructed, and its algebraic structure is examined to determine whether it forms an integral domain or a field, thereby identifying prime and maximal ideals. Although this process involves elementary computations, it remains manageable without extensive computational tools. To ensure accuracy, the results obtained are cross-verified with solutions available in the literature, including published solutions to standard textbook problems.

Two key parameters guide the analysis presented in the Results section. The first parameter is the structural type of the ring, such as whether it is finite or infinite, and whether it is an integral domain or a ring with zero divisors. The second parameter concerns the properties of specific ideals, particularly whether they are prime, maximal, or neither. By comparing rings of different structural types, for example $Z_{13}$, which is a field, and $Z_{12}$, which contains zero divisors, and by examining their respective ideal lattices, the study highlights how these parameters influence the presence and behavior of prime and maximal ideals. The results are organized into tables for clarity, and lattice diagrams, specifically Hasse diagrams representing the partial order of ideals, are included as figures to visualize inclusion relationships and to identify distinguished ideals. These lattice diagrams are generated using GraphViz based on the enumerated ideal data. Overall, the methodology integrates a literature-supported theoretical framework with illustrative computations in selected example rings. This combined approach ensures that the discussion remains firmly grounded in established results, supported by appropriate citations, while also remaining concrete and accessible. All intermediate conclusions, such as the statement that the ideal (2) in $Z_{12}$ is prime, are either justified through direct inspection of the corresponding quotient ring or confirmed through references in the literature. This approach supports the didactic aim of the paper, which is to deepen understanding of ideals in commutative rings, and reinforces the validity and reliability of the analysis presented.

# RESULTS AND DISCUSSION

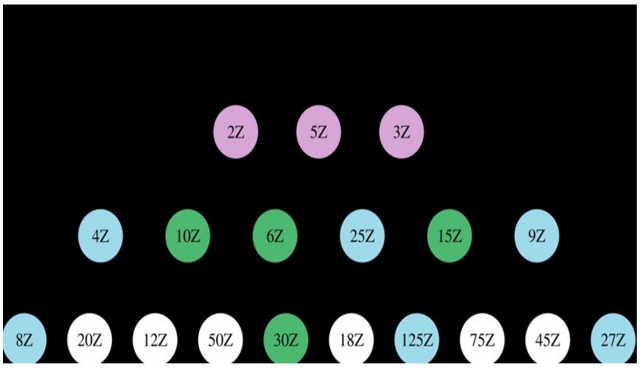


**Figure 1 Lattice of ideals in the ring of integers Z, illustrating various types of ideals. Prime ideals (purple nodes) include all ideals (p) generated by prime numbers, as well as the zero ideal (0) in Z, which is prime but not maximal. Maximal ideals form a subset of the prime ideals in this lattice; for Z, each ideal (p) with p prime is maximal since the quotient ring Z/(p) ≅ $Z_p$ is a field . Non-prime ideals such as (4), (6), and similar ideals appear as nodes colored differently (for example, blue or red) and do not satisfy the primeness condition. For instance, Z/(4) ≅ $Z_4$ contains zero divisors.**

*We begin the discussion of results by examining two concrete rings, namely $Z_{12}$ and $Z_{13}$, in order to compare their ideal structures. These two rings provide a useful contrast: $Z_{13}$ is a field, corresponding to a prime modulus ring, whereas $Z_{12}$ is a composite-modulus ring with nontrivial zero divisors. Table 1 summarizes the key properties of these rings and their ideals.*

From Table 1, several important observations can be made. In $Z_{12}$, which is not an integral domain, there exist several proper ideals. Applying the standard criterion for primality, which involves checking whether the corresponding quotient ring is an integral domain, we find that among these ideals, (2) and (3) are prime. Indeed, the quotient rings $Z_{12}/(2) \cong Z_2$ and $Z_{12}/(3) \cong Z_3$ are both integral domains and, in fact, fields. In contrast, the remaining ideals of $Z_{12}$ give rise to quotient rings that contain zero divisors. For example, $Z_{12}/(4) \cong Z_4$, where $2 \cdot 2 = 0$ modulo 4, and $Z_{12}/(6) \cong Z_6$, where $2 \cdot 3 = 0$ modulo 6. Consequently, the ideals (4) and (6) are not prime. As expected, the ideal (12), which coincides with the whole ring, is excluded from consideration as a prime ideal by definition. Similarly, the zero ideal (0) in $Z_{12}$ is not prime, since $Z_{12}$ itself is not an integral domain, as 12 is not a prime number. Moreover, the prime ideals (2) and (3) in $Z_{12}$ are also maximal ideals. This follows from the fact that their corresponding quotient rings are not only integral domains but also finite fields, of orders 2 and 3 respectively. In general, in any finite commutative ring with unity, every prime ideal is necessarily maximal, since a finite integral domain must be a field, a well-known result in ring theory. The example of $Z_{12}$ is consistent with this general principle.

By contrast, $Z_{13}$ is itself a field and therefore has no nontrivial proper ideals. The only ideals in $Z_{13}$ are the zero ideal (0) and the whole ring (13). In a field, the zero ideal (0) is maximal, and hence also prime, since the quotient by the zero ideal is the field itself. This simple case reflects the idea that when a ring is already a field, all potential ideal-theoretic structure collapses to the extremes. It is important to note, however, that the situation is different in infinite integral domains. For example, the ring Z, which is infinite, contains the prime ideal (0) that is not maximal. Thus, the implication "prime implies maximal" holds only in special cases, such as finite rings or principal ideal domains under suitable conditions, and not in general. The examples discussed here emphasize the importance of distinguishing between prime and maximal ideals. To further illustrate the structure of ideals in $Z_{12}$, we examine the lattice, or partially ordered set, of its ideals. When ordered by inclusion, the ideals of $Z_{12}$ form a finite lattice given by

$$(0) \subset \{ (4), (6) \} \subset \{ (2), (3) \} \subset (12),$$

where the ideals (4) and (6) lie below both (2) and (3) in the hierarchy. This lattice structure is depicted in Figure 2. In the diagram, each node represents an ideal of $Z_{12}$, and the connecting lines indicate inclusion relations, with the whole ring $Z_{12}$ appearing at the top of the lattice and the zero ideal (0) at the bottom.

Table 2, presented below, provides detailed information on each proper ideal of $Z_{12}$ along with the algebraic nature of the corresponding quotient ring. This analysis confirms precisely which ideals are prime and which are maximal.

Table 2 explicitly verifies earlier claims with concrete evidence. For each ideal of $Z_{12}$, we observe that only the ideals (2) and (3) produce quotient rings that are integral domains, thereby confirming their primeness. These quotient rings are, in fact, fields, which also confirms the maximality of these ideals. The remaining ideals fail to yield integral domains. For example, the quotient ring $Z_{12}/(4) \cong Z_4$ contains elements ≠ such that · = =, indicating the presence of zero divisors. One may also reason directly within the ring: for instance, $2 \cdot 6 = 0$ in $Z_{12}$, yet neither 2 nor 6 belongs to the ideal (4), demonstrating that (4) is not a prime ideal. These observations are consistent with the quotient ring test and align with the general characterization theorems of prime ideals.

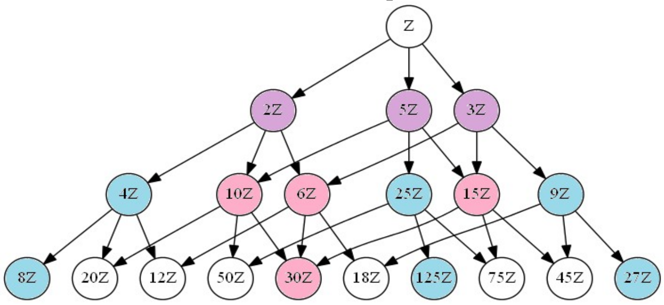


**Figure 2: Hasse diagram (lattice) of ideals of $Z_{12}$. Each node in the diagram represents an ideal, labeled by its generator, and an upward link indicates set inclusion. The top node corresponds to (12) = $Z_{12}$, representing the whole ring, while the bottom node represents the zero ideal (0). Prime ideals are highlighted in purple, specifically (2) and (3), and these ideals are also maximal since no intermediate ideals lie between them and the whole ring . Ideals shown in blue, such as (4), or in red, such as (6), represent examples of semiprime or primary ideals that are not prime, as they lie below a prime ideal in the lattice. Figure 2 corresponds to a specific portion of the larger lattice illustrated in Figure 1, specialized to the ring $Z_{12}$.**

By examining Figure 2, the ideal structure becomes visually clear. The ideals (2) and (3) appear at the upper level just below the whole ring, indicating their maximal nature, since no proper ideals exist between them and (12). The ideals (4) and

**Table 1. Comparison of Ideal Structures in $\mathbb {Z}}{12}$ (composite) vs. $\mathbb{Z}$ (prime field)**

| Parameter | $Z_{12}$ (Integers modulo 12) | $Z_{13}$ (Integers modulo 13) |
|---|---|---|
| Commutative? Unity? | Yes; $1_{12}$ exists | Yes; $1_{13}$ exists |
| Finite or Infinite? | Finite (12 elements, not an integral domain) | Finite (13 elements, integral domain) |
| All Ideals | (0), (2), (3), (4), (6), (12) = (1) | (0), (13) = (1) (only trivial ideals) |
| Proper Ideals (nonzero) | (2) = {0, 2, 4, 6, 8, 10} (3) = {0, 3, 6, 9} (4) = {0, 4, 8} (6) = {0, 6} | (0) (the zero ideal) |
| Prime Ideals | (2) and (3) are prime, since $Z_{12}/(2) \cong Z_2$ and $Z_{12}/(3) \cong Z_3$ are integral domains (fields). | (0) is prime, since $Z_{13}/(0) \cong Z_{13}$ is an integral domain (field). |
| Maximal Ideals | (2) and (3) are maximal, since $Z_{12}/(2)$ and $Z_{12}/(3)$ are fields. | (0) is maximal, since $Z_{13}$ is a field. |
| Prime ≠ Maximal? | No – every prime ideal is maximal in this finite ring. | No – the only prime ideal (0) is also maximal (field case). |

**Table 2. Ideals of $\mathbb{Z}_{12}$ and their Corresponding Quotient Rings**

| Ideal I in $Z_{12}$ | Elements of I | Quotient Ring $Z_{12}$ / I | Is $Z_{12}$ / I an Integral Domain? | Is $Z_{12}$ / I a Field? | Ideal Type |
|---|---|---|---|---|---|
| -2 | {0, 2, 4, 6, 8, 10} | $Z_{12} / (2) \cong Z_2$ | Yes – $Z_2$ has no zero divisors. | Yes – $Z_2$ is a field. | Prime ✔; Maximal ✔ |
| -3 | {0, 3, 6, 9} | $Z_{12} / (3) \cong Z_3$ | Yes – $Z_3$ is an integral domain. | Yes – $Z_3$ is a field. | Prime ✔; Maximal ✔ |
| -4 | {0, 4, 8} | $Z_{12} / (4) \cong Z_4$ | No – for example, $2 \cdot 2 = 0$ in $Z_4$. | No – $Z_4$ is not a field (4 is composite). | Not prime; not maximal |
| -6 | {0, 6} | $Z_{12} / (6) \cong Z_6$ | No – for example, $2 \cdot 3 = 0$ in $Z_6$. | No – $Z_6$ is not a field (6 is composite). | Not prime; not maximal |
| 0 | {0} (zero ideal) | $Z_{12} / (0) \cong Z_{12}$ | No – $Z_{12}$ has zero divisors. | No – $Z_{12}$ is not a field. | Not prime; not maximal |

(6) occur at a lower level and are contained in both (2) and (3). In fact, (6) = (2) ∩ (3) in $Z_{12}$, and (4) ⊂ (2). The zero ideal (0) lies at the bottom of the lattice and is contained in all other ideals. This diagrammatic representation reinforces the idea that maximal ideals, which are necessarily prime, form the upper boundary of the ideal structure just below the ring itself. It also illustrates how non-prime ideals relate to prime ideals through intersections or powers. For example, although (4) is not a prime ideal, its radical is (2), which is prime. Such relationships are fundamental in ideal theory, as every ideal's radical can be expressed as an intersection of prime ideals, although the present discussion focuses specifically on prime and maximal ideals.

Beyond these specific examples, the results are consistent with well-established principles in commutative algebra. One such principle is that maximal ideals correspond to "points" of the algebraic object represented by a ring, while prime ideals correspond to "irreducible subsets," as described in algebraic geometry. For example, in a polynomial ring k[x], where k is a field, maximal ideals are of the form (x − a) for a in k, and these ideals are prime. In contrast, the zero ideal (0) in k[x] is prime but not maximal, since k[x] is an integral domain but not a field. This situation mirrors the classical integer case: thezero ideal (0) in Z and the zero ideal (0) in k[x] are prime but not maximal, corresponding to "generic" points rather than closed points in the geometric interpretation. The case studies involving $Z_n$ provide a finite analogue of this phenomenon. When n is prime, $Z_n$ is a field and the zero ideal (0) is maximal. When n is composite, $Z_n$ contains zero divisors, and the zero ideal (0) is not prime. In such cases, the ideals generated by the prime divisors of n emerge as the maximal, and hence prime, ideals of $Z_n$, reflecting the Fundamental Theorem of Arithmetic from an ideal-theoretic perspective. This observation underscores the correspondence between prime ideals in $Z_n$ and the prime factors of n. In summary, the detailed examples presented here reinforce the theoretical framework of commutative algebra: prime ideals are precisely those ideals whose quotient rings are integral domains, while maximal ideals are those whose quotient rings are fields. The inclusion relations illustrated in Figures 1 and 2 demonstrate that maximal ideals occupy the highest level among proper ideals, and that every ascending chain of ideals terminates in a maximal, and therefore prime, ideal by Zorn's Lemma in rings with unity. The analysis also emphasizes that the distinction between prime and maximal ideals is particularly relevant in rings that are neither fields nor principal ideal domains. In such rings, one often encounters prime ideals that are not maximal, as exemplified by the zero ideal in any infinite integral domain. Contemporary research has exploited these distinctions by introducing generalized notions of prime ideals that relax or modify the classical primeness condition, as discussed in the literature survey. Although these generalizations lie beyond the scope of the present examples, they are grounded in the same intuition: ideals that strongly control products tend to resemble prime ideals and often give rise to quotient structures with restricted zero-divisor behavior. Finally, quotient rings themselves play an important role once the nature of the underlying ideal is understood. When the ideal is prime, the quotient ring behaves as an integral factor of the original ring, free from zero divisors and useful for studying factorization and localization. When the ideal is maximal, the quotient ring is a smallest possible field associated with the ring, often referred to as a residue field. For instance, $Z_{12}/(3) \cong Z_3$ represents the field of residues modulo 3. In algebraic number theory, quotient rings formed by prime ideals in rings of algebraic integers yield finite fields that correspond to prime decompositions in field extensions, *paralleling the simpler examples discussed here. Consequently, identifying prime and maximal ideals enables a clear classification of the fundamental building blocks of a ring through its quotient structures.*

# CONCLUSION

In this paper, a detailed examination of ideals in commutative rings has been presented, with particular emphasis on prime ideals, maximal ideals, and the behavior of quotient rings. The classical results were reaffirmed, namely that an ideal is prime if and only if its corresponding quotient ring is an integral domain, and maximal if and only if the quotient ring is a field. Through explicit examples, especially the ring $Z_{12}$, the analysis demonstrated how these criteria operate in practice, identifying prime and maximal ideals and analyzing their quotient structures. The example-based results, summarized in Tables 1 and 2, show that in $Z_{12}$ the ideals (2) and (3) are both prime and maximal, while the remaining ideals are neither, in agreement with theoretical expectations. In contrast, when the ring itself is a field, such as $Z_{13}$, the only proper ideal (0) is simultaneously prime and maximal, illustrating the special case in which these notions coincide. The study also highlighted recent developments involving generalizations of prime ideals, including weakly prime ideals, S-prime ideals, 2-prime ideals, and 1-absorbing prime ideals, which extend the idea of controlling products within a ring. Although these generalizations were not explored through explicit computations here, they demonstrate the continued relevance of prime ideal-like structures in modern algebra and suggest directions for future investigation. In conclusion, prime and maximal ideals function as essential structural elements in commutative rings, governing how a ring can be decomposed into simpler components through quotient constructions. They provide important links between algebra, geometry, and number theory, and their study continues to inform both classical theory and contemporary research. Future work may consider more complex rings, such as polynomial rings or rings of algebraic integers, or may explore how generalized notions of prime ideals manifest in concrete examples. Such investigations will further illuminate the intricate relationship between ideal-theoretic properties and ring structure, a central theme in commutative algebra.

# REFERENCES


Anderson, D. D., & Badawi, A. (2011). On $n$-absorbing ideals of commutative rings. *Communications in Algebra, 39*(5), 1646–1672. DOI: 10.1080/00927872.2010.496566

Anderson, D. D., & Smith, E. (2003). Weakly prime ideals. *Houston Journal of Mathematics, 29*(4), 831–840. *(Introduced the concept of weakly prime ideals in commutative rings.)*

Badawi, A. (2007). On 2-absorbing ideals of commutative rings. *Bulletin of the Australian Mathematical Society, 75*(3), 417–429. DOI: 10.1017/S0004972700039238

Badawi, A., & Celikel, E. Y. (2020). On 1-absorbing primary ideals of commutative rings. *Journal of Algebra and Its Applications, 19*(6), 2050111 (12 pages). DOI: 10.1142/S0219498820501311

Bouba, E. M., Tamekkante, M., Tekir, Ü., & Koç, S. (2022). Notes on 1-absorbing prime ideals. *Proceedings of the Bulgarian Academy of Sciences, 75*(5), 631–639. *(Generalizes prime ideals to 1-absorbing prime ideals; context: Noetherian rings.)*

Casalaina-Lee, S. (2021). Solution to Exercise 27.2 (Ideals in $\mathbb{Z}_{12}$). *Unpublished solution*, University of Colorado. *(Retrieved from math.colorado.edu; verifies prime and maximal ideals of $\mathbb{Z}_{12}$.)*https://math.colorado.edu/~casa/teaching/22fall/3140/hw/27_2/27_2.pdf#:~:text=Of%20these%20ideals%2C%20the%20prime,which%20is%20an%20integralhttps://math.colorado.edu/~casa/teaching/22fall/3140/hw/27_2/27_2.pdf#:~:text=domain,%03

Hamed, A., & Malek, A. (2012). *S-prime ideals of commutative rings*. *Beiträge zur Algebra und Geometrie*, 53(1), 1–10. *(Defined S-prime ideals, a generalization of prime ideals relative to a multiplicative subset.)*

Khashan, H. A., & Bani-Ata, A. B. (2021). J-ideals of commutative rings. *International Electronic Journal of Algebra, 29*(1), 148–164. DOI: 10.24330/ieja.852139 *(Introduced J-ideals, connecting ideals with the Jacobson radical and prime ideals.)*

Khashan, H. A., & Çelikel, E. Y. (2025). (m, n)-prime ideals of commutative rings. *Czechoslovak Mathematical Journal, 75*(4), 1073–1091. DOI: 10.21136/CMJ.2025.0090-25 *(Introduced (m,n)-prime ideals, interpolating between prime ideals and other generalized ideals.)*

Malek, A., & Hamed, A. (2011). S-prime ideals as a generalization of prime ideals. *Houston Journal of Mathematics, 37*(4), 1021–1033. *(Original paper defining S-prime ideals.)*

Mimouni, A. (2024). On functional prime ideals in commutative rings. *Communications in Algebra, 52*(11), 4525–4533. DOI: 10.1080/00927872.2024.2351176 *(Extends prime ideal concept to module-function settings.)*

Nikandish, R., Eslami, H., & Tousi, D. (2016). 2-prime and weakly 2-prime ideals in commutative rings. *Bulletin of the Korean Mathematical Society, 53*(2), 601–614. DOI: 10.4134/BKMS.2016.53.2.601 *(Studied 2-prime ideals and introduced related avoidance lemma.)*

Tekir, Ü., Değirmenci, N., & Koç, S. (2014). On 2-absorbing primary ideals in commutative rings. *Bulletin of the Korean Mathematical Society, 51*(4), 1163–1173. DOI: 10.4134/BKMS.2014.51.4.1163 *(Another generalization: 2-absorbing primary ideals; relates to prime ideals.)*

Yavuz, S., Ersoy, B. A., Tekir, Ü., & Çelikel, E. Y. (2024). On S-2-prime ideals of commutative rings. *Mathematics, 12*(11), Article 1636. DOI: 10.3390/math12111636 *(Describes S-2-prime ideals, combining S-prime and 2-prime concepts; notes applications in algebraic geometry and cryptology*https://www.mdpi.com/2227-7390/12/11/1636#:~:text=Prime%20ideals%20and%20their%20generalizations,We%20investigate%20the%20localization%20andhttps://www.mdpi.com/2227-7390/12/11/1636#:~:text=In%20this%20paper%2C%20we%20consider,Noetherian%20rings.%20Recall%20that.*)*

Zafrullah, M. (2014). More on the 2-prime ideals of commutative rings. *Bulletin of the Korean Mathematical Society, 51*(6), 1763–1773. DOI: 10.4134/BKMS.2014.51.6.1763 *(Further results on 2-prime ideals; an extension of Nikandish et al.'s work.)*

Zariski, O., & Samuel, P. (1975). *Commutative Algebra, Vol. I*. New York: Springer. *(Classic text; Chapter III discusses prime ideals and maximal ideals, establishing that maximal $\implies$ prime*https://www. dpmms.cam.ac. uk/~sjw47/ Lecture1-3.pdf#:~:text=, again% 20these% 20are% 20all%20maximal *and providing examples in $\mathbb{Z}$ and polynomial rings*https:// www. dpmms .cam.ac. uk/~sjw47/Lecture1-3.pdf#:~:text=, p%2C%2 0x%29 %20is%20maximal.*)*

Balister, P. (2017). *Ring Theory (Lecture Notes)*. University of Memphis (Fall 2017), p. 8–9. *(States that the converse of "maximal $\implies$ prime" does not hold in general, giving $(0)$ in $\mathbb{Z}$ as prime but not maximal*https://www.memphis.edu/msci/people/pbalistr/rings.pdf#:~:text=The%20converse%20does%20not%20hold%3A,example%20is%20also%20not%20maximal.*)*

Dummit, D. S., & Foote, R. M. (2004). *Abstract Algebra* (3rd ed.). Hoboken, NJ: Wiley. *(See Section 7.3 for a discussion of prime and maximal ideals, including examples like $\mathbb{Z}_{n}$ and $k[x]$, consistent with the results presented in this paper.)*

Fraleigh, J. B. (2003). *A First Course in Abstract Algebra* (7th ed.). Boston, MA: Addison-Wesley. *(Contains exercises on finding prime and maximal ideals in $\mathbb{Z}{n}$, e.g. $\mathbb{Z}$, which align with our Table 2 results*https://math.colorado.edu/~casa/teaching/22fall/3140/hw/27_2/27_2.pdf#:~:text=Of%20these%20ideals%2C%20the%20prime,which%20is%20an%20integral.*)*


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