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RESEARCH ARTICLE

EXPECTED NUMBER OF LEVEL CROSSING OF A RANDOM ORTHOGONAL POLYNOMIAL

***Pralipta Rout and Dr. P.K Mishra**

Department of Mathematics, Odisha University of Technology and Research, Techno Campus Mahalaxmi Vihar, Nakagate, Khandagiri, Bhubaneswar, Odisha

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**Corresponding author:*
Pralipta Rout

ABSTRACT

Let T_0^*(x), T_1^*(x)……….,T_n^*(x) be a sequence of a normalized Legendre polynomials orthogonal with respect to the interval $(-1,1)$. This paper provides an asymptotic estimate for the expected number of K-level crossings of the random polynomial $g_0T_0^*$(x)+$g_1T_1^*$(x)+……….+$g_nT_n^*$(x) where g_j(j = 0,1,…………,n) are independent normally distributed random variableswith mean zero and variance one. It is shown that the result for K = 0 remains valid for any K such that $(K^2/n)\rightarrow 0$ as n → ∞.

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INTRODUCTION

Let $\quad$ P(x) ≡ P_n(x,w) = $\sum_{n=0}^{n} g_j(w)\, T_j^*(x)$ $\qquad$ (1.1)

Where g_1(w), g_2(w),…….,g_n(w) is a sequence of independent random variables defined on a probability space (Ω,A,Pr), each normally distributed with mathematical expectation zero and variance one. Let $N_K(\alpha,\beta) \equiv N(\alpha,\beta)$ be the number of real roots of the equation P(x) = K in the interval (α,β) where multiple roots are counted only once. For the different forms of $T_j{}^*$(x) asymptotic values for the mathematical expectation of N(α,β), denoted by $EN(\alpha,\beta)$, have been studied by various authors. Assuming T_j^*(x) = x^j and K = 0 it is shown, for example see Kac [7], that EN$(-\infty,\infty)$ $\sim (2/\pi)\log n$ for all sufficiently large n. This asymptotic value persists in the work of Offord [4] when they considered the discrete coefficients of having values +1 and $-$ 1 with equal probability. Farahmad [5] for the case of normal standard coefficients shows that for K ≠ 0 in the interval $(-1,1)$ the expected number of K level crossings i.e. roots of P(x) = K, is reduced to $(1/\pi)\log(n/K^2)$ while outside this interval this expected number remains the same as for the case of K = 0, as long as K = O$(\sqrt{n})$. For $T_j^*(x)$ = $\cos jx$, from the work of Dunnage [3] and Farahmand [6], we know that for any K = O$(\sqrt{n})$ and all sufficiently large n, EN(0,2π) ~ $(2/\sqrt{3})$n. Therefore by increasing it is invariant for the trigonometric one.

Here we consider the case of

$T_j^*(x)$=$\sqrt{(j+1/2)}T_j(x)$ $\qquad$ (1.2) where $T_j(x)$ is a Legendre polynomial, and therefore $T_j(x)$ is a normalized polynomial orthogonal with respect to the weight function unity. For K = 0 from Das [2] we know that EN$(-1,1) \sim (n/\sqrt{3})$ when n is sufficiently large. Now this is interesting as it raises the question as to which of the above patterns, if any the K-level crossings of the Legendre polynomial will follow, or what is equivalent , for any K = O$(\sqrt{n})$, where EN would reduce as K increase or not. As the oscillatory nature of classical orthogonal polynomial is accurately known we will show how far these oscillations are transformed into random sum (1.1), where $T_j(x)$ is defined as (1.2). We prove the following theorm:

THEORM 1. For any sequence of constants K_n such that (K^2/n) tends to zero as n tends to infinity, the mathematical expectation of the number of real roots of the equation T(x) = K, satisfies $EN(-1.1) \sim (n/\sqrt{3})$.

From the theorem therefore, we can see that, as far as the K- level crossings go, the behaviour of random Legendre polynomials is similar to that of trigonometric polynomials, that is unlike the algebraic case, the expected number of K-level crossings is invariant for any K = O$(\sqrt{n})$. On the basis of this evidence it seems interesting to ask, in general whether we can classify the oscillation of different types of polynomials according to the behaviour of their K-level crossings namely, the algebraic types with EN = O$(\log n)$ and the trigonometric types with EN = O(n).

2. APPROXIMATIONS

Let $\varphi(t) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \exp(-y^2/2)dt$ And
$\varphi(t) = \frac{d\varphi(t)}{dt} = (2\pi)^{-1/2} \exp(-t^2/2);$

Then by using the expected number of level crossings given by Cramer and Lead better [1, page 285] for our equation P(x)-K = 0 we can obtain

$EN(\alpha,\beta) = \int_{\alpha}^{\beta} (B/A)(1 - C^2/A^2B^2)^{1/2}\, \varphi(-K/A)\big(2\varphi(\eta)\big) + \eta\{2\varphi(\eta) - 1\}dx,$
Where A^2 = var$\{P(x)\}$, B^2 = var$\{P'(x)\}$ C = cov$\{P(x), P'(x)\}$

And

$\eta = -CK/A(A^2B^2 - C^2)^{1/2}.$
Let $\Delta^2 = A^2B^2 - C^2$ and erf(x) = $\int_0^x \exp(-t^2)dt;$ then we can write the extension of a formula obtained by Kac [7] and Rice [8] for the cae of K = 0 as

$EN(\alpha,\beta) = \int_{\alpha}^{\beta} \frac{\Delta}{\pi a^2} \exp\left(-\frac{B^2K^2}{2\Delta^2}\right)dx + \int_{\alpha}^{\beta} \frac{\sqrt{2|KC|}}{\pi A^3} \exp\left(-\frac{K^2}{2A^2}\right) erf\left(\frac{|KC|}{A\Delta\sqrt{2}}\right)dx$
$= I_1(\alpha,\beta) + I_2(\alpha,\beta)$,say. (2.1)

For our case of random Legendre polynomials we set

$R_{ij}(x) = T_{n+1}^{(i)}(x)T_n^{(j)}(x) - T_{n+1}^{(j)}(x)\, T_n^{(i)}(x)$ i = 0,1,2,3; j = 0,1,

Where $T_n^{(i)}(x)$ represents the ith derivative of $T_n(x)$ with respect to x. Then from the Darboux-Christoffel formula [7] putting
$\lambda_n = (n+1)(2n+3)^{1/2}/2(2n+1)^{1/2}.$

We can obtain

$$\sum_{j=0}^{n} \{T_j(x)\}^2 = (\lambda n)R_{10}(x), \tag{2.2}$$

$$\sum_{j=0}^{n} T_j^*(x)T_j^{*'}(x) = (\lambda_n/2)R_{20}(x) \tag{2.3}$$

And

$$\sum_{j=0}^{n} \{T_j^{*'}(x)\}^2 = (\lambda_n/6)R_{30}(x) + (\lambda_n/2)R_{21}(x). \tag{2.4}$$

We recall two well known recurrence formulae for Legendre polynomials [7],

$$nT_{n-1}(x) = (2n+1)xT_n(x) - (n+1)T_{n+1}(x) \tag{2.5}$$

and

$$(1-x^2)T_n'(x) = n\{T_{n-1}(x) - xT_n(x)\} \tag{2.6}$$

We rewrite (2.6) for $T'_{n+1}(x)$and by the application of (2.5) we can obtain

$$R_{10}(\mathrm{x}) = (n+1)\frac{T_{n+1}^2(x)+T_n^2(x)-2xT_n(x)T_{(n+1)}(x)}{1-x^2} \tag{2.7}$$

And

$$T''_{n+1}(x)T_n(x) + T_{n+1}(x)T'_n(x) = (n+1)\frac{T_n^2(x)-T_{n+1}^2(x)}{1-x^2} \tag{2.8}$$

To evaluate the right hand side of (2.7) we assume $-1+\varepsilon < x < 1-\varepsilon$ where ε is any positive value smaller than one and we set x = $\cos\gamma$. Then since from the Laplace formula [11] we have

$$T_n(\cos\gamma) = \sqrt{\frac{2}{n\pi\,\mathrm{si}}}\cos\left\{\left(n+\frac{1}{2}\right)\gamma-\frac{\pi}{4}\right\} + O\{(n\sin\gamma)^{-3/2}\}$$

We can obtain,

$$T_{n+1}^2(x) + T_n^2(x) - 2xT_n(x)T_{n+1}(x) = \frac{2}{n\pi\sin}\left[cos^2\left\{\left(n+\frac{1}{2}\right)\gamma-\frac{\pi}{4}\right\} + cos^2\left\{\left(n+\frac{3}{2}\right)\gamma-\frac{\pi}{4}\right\} - 2\cos\gamma cos\left\{\left(n+\frac{1}{2}\right)\gamma-\frac{\pi}{4}\right\} + cos^2\left\{\left(n+\frac{3}{2}\right)\gamma-\frac{\pi}{4}\right\}\right] + O(n\sin\gamma)^{-2}$$

$$= \frac{2\sqrt{1-x^2}}{n\pi} + O\left(\frac{1}{n^2(1-x^2)}\right) \tag{2.9}$$

Hence from (2.2), (2.3), (2.7), and (2.9) we get

$$A^2 = \frac{(n+1)^2(2n+3)^{1/2}}{n\pi(2n+1)^{1/2}(1-x^2)^{1/2}} + O\left(\frac{1}{n^2(1-x^2)^2}\right) \tag{2.10}$$

To evaluate B^2 and C we make use of the property that any Legendre polynomial T_n(x) satisfies the equation

$$(1-x^2)\frac{d^2u}{dx^2} - 2\mathrm{x}\frac{du}{dx} + \mathrm{n(n+1)u} = 0$$

This gives the value of T''_n(x) as

$$\frac{2xT'_n(x) - n(n+1)T_n(x)}{1-x^2}$$

Rewriting the above formula for $T'_n(x)$as well and then distributing them in the formulae for $R_{21}(x)$ and $R_{20}(x)$ to obtain

$$R_{21}(x) = \frac{-(n+1)\{nR_{01}(x)+2T_{n+1}(x)T'_n(x)\}}{1-x^2} \tag{2.11}$$

And

$$R_{20}(x) = \frac{2xR_{10}(x)-2(n+1)T_n(x)T_{n+1}(x)}{1-x^2} \tag{2.12}$$

Differentiating (2.12) and using (2.11) we get

$$R_{30}(x) = \frac{(n+1)\{nR_{01}(x)+2T'_{n+1}(x)T_n(x)+2R_{10}(x)\}}{1-x^2} + \frac{8x\{nR_{01}(x)-(n+1)T_n(x)T_{n+1}(x)\}}{(1-x^2)^2} \tag{2.13}$$

Now by the first theorm of Stielzer [11, page 197] we have $|T_n(x)| \le 8n^{1/2}(1-x^2)^{-5/4}$.

Thus

$T_n(x)T_{n+1}(x) = \mathrm{O}\left(\frac{1}{n(1-x^2)^{1/2}}\right)$ And

$$T_n(x)T'_n(x) = \mathrm{O}\left(\frac{1}{(1-x^2)^{3/2}}\right)$$

By putting these estimates in (2.11), (2.12) and (2.13) we can obtain

$$R_{21}(x) = \frac{n(n+1)R_{10}(x)}{(1-x^2)} + \mathrm{O}\left(\frac{n}{(1-x^2)^{5/2}}\right),$$

$R_{20}(x) = \frac{2xR_{10}(x)}{(1-x^2)} + O\left(\frac{1}{(1-x^2)^{3/2}}\right)$ And

$R_{21}(x) = \frac{\{8x^2/(1-x^2)+n+n^2\}R_{10}(x)}{(1-x^2)} + O\left(\frac{n}{(1-x^2)^{5/2}}\right),$

Substituting the above formulae in (2.3) and (2.4) and since from (2.7) and (2.9), $R_{10}(x) = 2(n+1)/n\pi(1-x^2)^{1/2}$ we get

$$C = O\left(\frac{n}{(1-x^2)^{3/2}}\right) \tag{2.14}$$

And

$$B^2 = \frac{(n+1)^3(2n+3)^{1/2}}{3\pi(2n+1)^{1/2}(1-x^2)^{3/2}} + O\left(\frac{n^2}{(1-x^2)^{5/2}}\right) \tag{2.15}$$

3. PROOF OF THE THEORM

From (2.1), (2.10), (2.14) and (2.15) we note that changing x to -x will not change $EN(\alpha,\beta)$. Therefore it suffices to determine the asymptotic behaviour of $EN(0,1)$. To this end we divide the real roots into two groups: (i) those lying in the interval $(0,\varepsilon)$ and $(1-\varepsilon,1)$ and (ii) those lying in the interval $(\varepsilon,1-\varepsilon)$. For the roots (i) which, it so happens, are negligible, we need some modification to apply Dunnage's [3] approach, which is based on an application of Jensen's theorm [10, page 332] or [12, page 125]. For roots (ii) which yield the main contribution to the expected number of real roots, we use (2.1). The ε should be chosen such that it facilitates handling type (i) and type (ii) roots and also yields the smallest possible error term in approximations . It is shown that $\varepsilon = n^{-1/4}$ satisfies both requirements.

CONCLUSION

In this paper, considering T_0^*(x), T_1^*(x)……….., T_n^*(x) be a sequence of a normalized Legendre polynomials orthogonal with respect to the interval $(-1,1)$. It provides an asymptotic estimate for the expected number of K-level crossings of the random polynomial $g_0T_0^*$(x)+$g_1T_1^*$(x)+………..+$g_nT_n^*$(x) where g_j(j = 0,1,…………,n) are independent normally distributed random variables with mean zero and variance one. The result for K = 0 remains valid for any K such that (K^2/n)→ 0 as n → ∞.

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