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RESEARCH ARTICLE

EXISTENCE OF (Φ⊗Ψ) BOUNDED SOLUTIONS FOR LINEAR SYSTEM OF FIRST ORDER DIFFERENTIAL EQUATIONS INVOLVING KRONECKER PRODUCT OF MATRICES

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ABSTRACT

This paper presents a criteria for the existence of (Φ⊗ Ψ) bounded solutions of linear first order system of differential equations involving Kronecker Product of Matrices.

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INTRODUCTION

In this paper, we shall be concerned with the existence of (Φ⊗ Ψ) bounded solutions of the two first order linear systems of differential equations of the form

$$x'(t) = A(t)x(t) + f(t), \quad a \le t \le b \qquad (1.1)$$

And

$$y'(t) = B(t)y(t) + g(t), \; a \le t \le b \qquad (1.2)$$

where $A(t)$, $B(t)$ are square matrices of orders $m \times m$ and $n \times n$ respectively, f and g are $(m \times 1)$ and $(n \times 1)$ column vectors and all scalars are assumed to be real. System of equations (1.1) and (1.2) can be conveniently be recasted as

$$[x(t) \otimes y(t)]' = [(A(t) \otimes I_n) + (I_m \otimes B(t))][x(t) \otimes y(t)] + [f \otimes I_n + I_m \otimes g](t), \quad a \le t \le b \qquad (1.3)$$

We shall be concerned with the existence of (Φ⊗ Ψ) bounded solutions of the linear Kronecker Product system of equation (1.3) and deduce the existing results as a particular case of our results. The paper is organized as follows: Section 2 presents certain interesting results on Kronecker product of matrices and use them as a tool to establish our main results mainly, the existence of (Φ⊗ Ψ) bounded solutions. The results on stability of linear continuous Kronecker product system are established in section 3. The results established in this section are generalized to Time scale dynamical system at the end of the section. Section 4 presents our main result. We now present the basic results on Kronecker product of matrices and their applications on linear system. We also present some basic notions on Time scale dynamical system in the next section.

Basic Results

Definition 2.1: Let $A = (a_{ij})$ be an $(m \times n)$ matrix and $B = (b_{ij})$ be a $(p \times q)$ matrix then their Kronecker product $(A \otimes B)$ is defined as

$(A \otimes B) = (a_{ij}B)$ for all $i = 1,2, \dots, m; j = 1,2, \dots, n$

and is in fact an $(mp \times nq)$ matrix. The kronecker product of matrices defined above has the following properties:

1. $(A \otimes B)(C \otimes D) = (AC \otimes BD)$ (provided AC and BD are defined)
2. $(A \otimes B)^T = A^T \otimes B^T$ (A^T stands for transpose of the matrix A)
3. $(A + B) \otimes C = (A \otimes B) + (B \otimes C)$
4. $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$ (provided A and B are invertible)
5. $(A \otimes B)(n) = (A(n) \otimes B(n))$ (For discrete systems) $(A \otimes B)(n+1) = (A(n+1) \otimes B(n+1))$ (For discrete systems)
6. $\|A \otimes B\| = \|A\|\|B\|$

Time Scale Dynamical System:

We now present, some of the basic properties on Time scale dynamical system and use them as a tool to establish our main results in the final section. By a time scale T to be a closed subset of R, and examples of time scales include, N, Z, R, Contour set, Fuzzy sets, Topological sets etc. The set

$Q = \{t \in R - Q, 0 \leq t \leq 1\}$ are not time scales. Time scales need not necessarily connected. In order to overcome this deficiency, we introduce the notion of Jump operators defined here under:

The operator $\sigma\,(t) = Inf\{t \in T : s > t\}$

$\rho\,(t) = Sup\{t \in T : s < t\}$

are called Jump operators. If σ is bounded above and ρ is bounded below, then we define

$\sigma\,(\max T) = \max T$, $\sigma\,(\min T) = \min T$

A point $t \in T$ is said to be right dense, if $\sigma\,(t) = t$, right scattered if $\sigma\,(t) > t$, left dense if $\rho\,(t) = t$ and left scattered, if $\rho\,(t) < t$.

The graininess μ: $[0, \infty)$ is defined as

$\mu(t) = \sigma\,(t) - t$.

We say that f is rd-continuous, if it is continuous at right dense points and if lim $f(s)$ as $s \to t$ exists at all right dense points $t \in T$.

A function $f: T \to T$ is said to be differentiable at $t \in T^{\mathrm{k}}$ if

$$\lim_{t \to s} \frac{f(\sigma\,(t)) - f(s)}{\sigma\,(t) - s}$$

exists and is said to be differentiable for each $t \in T^{\mathrm{k}}$

A function $f: T \to T$ with $F^{\Delta}(t) = f(t)$ for all $t \in T^{\mathrm{k}}$ is said to integrable, if

$\int_s^t f^{\Delta}(c) = F(t) - F(s)$,

where f is the anti-delta derivative of F for all $s, t \in T$

Let $f: T \to T$ and if $T = R$, $a, b \in T$ then $F^{\Delta}(t) = f(t)$ and $\int_a^b f(t)\Delta t = \int_a^b f(t)dt$,

and if $T = Z$, then $F^{\Delta}(t) = \Delta f(t) = f(t+1) - f(t)$ and

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{k=0}^{b-1} f(k), if\ a < b \\ 0,\ if a = b \\ \sum_{k=a}^{a-1} f(k), if\ a > b \end{cases}$$

Note that, if f is Δ-differentiable, then f is continuous. Further if it is right scattered and f is continuous at t, then

$$f^{\Delta}(t) = \frac{f(\sigma\,(t)) - f(s)}{\mu(t)}$$

For a complete survey on Δ- differentiable functions, we refer to an excellent survey made by M. Bohner and Allan Peterson [2].

Definition 2.2: A matrix P is said to be a projection matrix, if $P^2 = P$. If P is the projection matrix, then $I - P$ is also a projection. Two such projections whose sum is I and whose product is zero, are said to be complementary.

For any $(k \times k)$ real matrix $A = (a_{ij})$, $i, j = 1,2, \dots \dots, k$, we define the norm of the matrix A as

$\|A\| = \sup_{\|x\| \leq 1} \|Ax\| = |A|$

Definition 2.3: A mapping $f: T \to X$, where $X$is a Banach space is called rd-continuous if it is continuous at each right dense points $t \in T$.

If t is right scattered and f is continuous at t, then

$$f^{\Delta}(t) = \frac{f(\sigma\,(t)) - f(t)}{\mu(t)}$$

where $\mu(t) = \sigma\,(t) - t$

Definition 2.4: Any set of n-linearly independent solutions $y_1, y_2, \dots, y_n$ of

$y^{\Delta}(t) = A(t)y(t)$

is called a fundamental set and the matrix with$y_1, y_2, \dots, y_n$is its columns is called a fundamental matrix and is denoted by Y. We now present an interesting result which is very useful in our future discussion and in fact focus a basis for establishing our main result.

Result 2.1:

Let $\Phi(t)$ is a fundamental matrix of $x'(t) = A(t)x(t)$ and $\Psi(t)$ is a fundamental matrix of $y'(t) = B(t)y(t)$, where $A(t)$ and $B(t)$ are square matrices of orders $m \times m$ and $n \times n$ respectively. Then $(\Phi(t) \otimes \Psi(t))$ is a fundamental matrix of $[x(t) \otimes y(t)]' = [(A(t) \otimes I_n) + (I_m \otimes B(t))][x(t) \otimes y(t)]$ if and only if

$\Phi(t)$ is a fundamental matrix of $x'(t) = A(t)x(t)$ and $\Psi(t)$ is a fundamental matrix of $y'(t) = B(t)y(t)$

Proof: Suppose $\Phi(t)$ and $\Psi(t)$ be a fundamental matrices of

$x'(t) = A(t)x(t)$ and $y'(t) = B(t)y(t)$ then consider

$$[\Phi(t) \otimes \Psi(t)]' = \left[\left(\Phi'(t) \otimes \Psi(t)\right) + \left(\Phi(t) \otimes \Psi'(t)\right)\right]$$
$$= [A(t)\Phi(t) \otimes \Psi(t)] + \left(\Phi(t) \otimes B(t)\Psi(t)\right)$$
$$= \left[(A(t) \otimes I_n) + \left(I_m \otimes B(t)\right)\right][\Phi(t) \otimes \Psi(t)]$$

Hence $[\Phi(t) \otimes \Psi(t)]$is a fundamental matrix of

$$[x(t) \otimes y(t)]' = \left[(A(t) \otimes I_n) + \left(I_m \otimes B(t)\right)\right][x(t) \otimes y(t)] \qquad (2.1)$$

Conversely, suppose $(\Phi(t) \otimes \Psi(t))$ is a fundamental matrix of (2.1), then it is claimed that, $\Phi(t)$ is a fundamental matrix of $x'(t) = A(t)x(t)$ and $\Psi(t)$ is a fundamental matrix of $y'(t) = B(t)y(t)$

For, consider

$$[\Phi(t) \otimes \Psi(t)]' = \left[\left(\Phi'(t) \otimes \Psi(t)\right) + \left(\Phi(t) \otimes \Psi'(t)\right)\right]$$
$$= \left[(A(t) \otimes I_n) + \left(I_m \otimes B(t)\right)\right][\Phi(t) \otimes \Psi(t)]$$

Then

$$\left[\Phi'(t) - A(t)\Phi(t)\right] \otimes \Psi(t) = \Phi(t) \otimes [B\Psi(t) - \Psi'(t)]$$

Multiplying both sides of the above relation of $(\Phi^{-1}(t) \otimes \Psi^{-1}(t))$, we get

$$\Phi^{-1}(t)\left[\Phi'(t) - A(t)\Phi(t)\right] \otimes I_n = I_m \otimes \Psi^{-1}(t)[B\Psi(t) - \Psi'(t)] \qquad (2.2)$$

The above relation holds if and only if each of the bracketed expression of (2.2) is either a null matrix or a unit matrix. Let us examine each case separately. Suppose each side is a null matrix. Then

$$\Phi^{-1}(t)\left[\Phi'(t) - A(t)\Phi(t)\right] = O$$

Hence

$$\left[\Phi'(t) - A(t)\Phi(t)\right] = O$$

Hence $\Phi(t)$ is a fundamental matrix of $x'(t) = A(t)x(t)$. Similar argument holds for the other bracketed expression on the right side of (2.2)

Suppose each side of (2.2) is a unit matrix. Then

$$\Phi^{-1}(t)\left[\Phi'(t) - A(t)\Phi(t)\right] = I_m$$
$$\Phi'(t) - A(t)\Phi(t) = \Phi(t)$$
$$\Phi'(t) = [A(t) + I_m]\Phi(t)$$

which is a contradiction. Similarly the other part.

Similar result holds for Time scale dynamical systems of the form

$$x^{\Delta}(t) = A(t)x(t) \qquad (2.3)$$

$$y^{\Delta}(t) = B(t)y(t) \qquad (2.4)$$

(2.3) and (2.4) can be uniquely recasted as

$$(x(t) \otimes y(t))^{\Delta} = [(A(t) \otimes I_n) + \left(I_m \otimes B(t)\right)][x(t) \otimes y(t)] \quad (2.5)$$

Let $\Phi(t)$ is a fundamental matrix solution of the time scale dynamical system (2.3) and $\Psi(t)$ is a fundamental matrix solution of (2.4). Then $(\Phi(t) \otimes \Psi(t))$ is a fundamental matrix of (2.5), if and only if $\Phi(t)$ is a fundamental matrix of (2.3) and $\Psi(t)$ is a fundamental matrix of (2.4).

Stability Analysis of Kronecker Product systems: It is a well known fact that the word stability, we mean the ability of the system to come back to its natural position after a small perturbance. If a small perturbance in the initial data results a substantial deviation from the original behavior of solution, then such a solution is not acceptable even approximately. The process of finding conditions under which such deviations will not happen is vital in any branch of Science and Engineering. So we investigate conditions under which the solutions of Kronecker produt system is stable, asymptotically stable.

Theorem 3.1: Let $\Phi(t)$ is a fundamental matrix of $x'(t) = A(t)x(t)$ and $\Psi(t)$ is a fundamental matrix of $y'(t) = B(t)y(t)$. Then the Kronecker product system (1.3) is stable if and only if there exists a constant $M > 0$ such that $\|\Phi(t)\| \leq M$ and there exists a constant $K > 0$ such that $\|\Psi(t)\| \leq K$, for all $t \geq 0$

Proof: Suppose $\|\Phi(t)\|$ and $\|\Psi(t)\|$ are bounded then

$$\|\Phi(t) \otimes \Psi(t)\| \leq \|\Phi(t)\|\|\Psi(t)\|$$
$$\leq \mathrm{MK}$$

Hence the system is stable. Suppose the system $x'(t) = A(t)x(t)$ is stable. Then it can be easily be proved that its fundamental matrix is bounded and similarly the fundamental matrix $\Psi(t)$ is bounded.

Theorem 3.2: Suppose a fundamental matrix of $x'(t) = A(t)x(t)$ is such that $\|\Phi(t)\| \to 0$ as $t \to \infty$. Then the system $x'(t) = A(t)x(t)$ is said to be asymptotically stable.

Theorem 3.2: Suppose either $\Phi(t)$ or $\Psi(t)$ of the respective systems is asymptotically stable. Then the Kronecker product system (1.3) is asymptotically stable, provided both $\Phi(t)$ and $\Psi(t)$ are bounded.

Proof: suppose $\|\Phi(t)\|$ is bounded and $\Psi(t) \to 0$ as $t \to \infty$. Then consider

$\|\Phi(t) \otimes \Psi(t)\| \leq M\|\Psi(t)\|$ where $\|\Phi(t)\| \leq \mathrm{M}$

If $\Psi(t) \to 0$ as $t \to \infty$. Hence the Kronecker product system (1.3) is asymptotically stable. Similarly, if $\Psi(t)$ is bounded then $\|\Phi(t) \otimes \Psi(t)\| \to 0$ as $t \to \infty$ and hence (1.3) is asymptotically stable. Similar results hold for system on time scales. In order to avoid monotony, we even omit stating those results.

MAIN RESULTS

In this section, we shall concerned with the existence of $(\Phi \otimes \Psi)$ bounded solution of the linear Kronecker system (1.3). Further we assume that the homogeneous system admits at least one $(\Phi \otimes \Psi)$ bounded solution for every Lebesgue $(\Phi \otimes \Psi)$ delta integrable function.

Further, we suppose that the system (1.1) admits a Φ bounded solution satisfying the initial condition $x(0) \in X_0$ and Ψ bounded solution on $T^+ = [0, \infty)$ if and only if $x(0) \in X_- \otimes X_0$, where X_-, X_0 are two subspaces such that a solution $y(t)$ of (1.2) is Φ bounded and a solution $x(t)$ of (1.1) is Ψ bounded. Further, if X_-, X_0, X_+ are such that $x(0) \in X_- \otimes X_0$ and $y(0) \in X_0 \otimes X_+$. Theorem 4.1: Let A and Bbe continuous $m \times m$ and $n \times n$ matrices, then (1.3) has at least one $(\Phi \otimes \Psi)$ bounded solution on T for every Lebesgue β-Δ integrable function if and only if there exists a positive constant $K > 0$ such that

(i) $\|\alpha(t)\beta(t)P_-(\sigma(s))\alpha^{-1}(\sigma(s))\| \leq K$, for $t > 0$, $\sigma(s) < 0$
(ii) $\|\alpha(t)\beta(t)(P_0 + P_+)\beta^{-1}(\sigma(s))\alpha^{-1}(\sigma(s))\| \leq K$, for $t > 0, \sigma(s) < 1$
(iii) $\|\alpha(t)\beta(t)P_+\beta^{-1}(\sigma(s))\alpha^{-1}(\sigma(s))\| \leq K$, for $t > 0$, $\sigma(s) > 0$,
$\sigma(s) > t$
(iv) $\|\alpha(t)\beta(t)(P_-)\beta^{-1}(\sigma(s))\alpha^{-1}(\sigma(s))\| \leq K$, for $t \leq 0$, $\sigma(s) < t$
(v) $\|\alpha(t)\beta(t)(P_0 + P_+)\beta^{-1}(\sigma(s))\alpha^{-1}(\sigma(s))\| \leq K$, for $t \leq 0, \sigma(s) \geq t$
$\sigma(s) < 0$
(vi) $\|\alpha(t)\beta(t)P_+\beta^{-1}(\sigma(s))\alpha^{-1}(\sigma(s))\| \leq K$, for $t \leq 0$, $\sigma(s) \geq t$,
$\sigma(s) \geq 0$

Proof: First suppose that the system (1.3) has at least one α-bounded solution on T for every Lebesgue α-Δ integrable function $f: T \to T$.

Then define

$\mathcal{C}_\alpha$: The Banach space of all α-bounded continuous functions $x: T \to T$

with norm

$\|x\|_{\mathcal{C}_\alpha} = Sup_{t \in T}\|\alpha(t)x(t)\|$,

$\mathcal{B}$: The Banach space of all α –Δ integrable functions $x: T \to T$with the norm

$$\|x\|_{\mathcal{B}} = \int_{-\infty}^{\infty} \|\alpha(t)x(t)\| \, \Delta\text{t}$$

$\mathcal{D}$: The set of all functions $x: T \to T$ which are absolutely continuous functions on all real intervals $J \subset T\alpha$-bounded on T, $x(0) \in X_- \otimes X_+$ and $y(0) \in X_0 \otimes X_+$. It can easily proved that $(\mathcal{D}, \|.\|_{\mathcal{D}})$ is a Banach space and further there exists a positive constant $K > 0$ such that for every $f \in B$ and for every corresponding solution $\hat{x} \in DT\hat{x} = \left[x^\Delta - \left((A \otimes I_n) + (I_m \otimes B)\right)\right] \hat{x}$

The remaining of the proof of the theorem follows as in [6].

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