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RESEARCH ARTICLE

COMMON FIXED POINTS OF F-CONTRACTION MAPPING WITH GENERALIZED ALTERING DISTANCE FUNCTION IN PARTIALLY ORDERED METRIC SPACES

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ABSTRACT

The concept of anf-contraction mapping with generalized altering distance function is introduced, and some fixed and common fixed point theorems for f-contraction mapping with generalized altering distance function in partially ordered metric spaces are proved.

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1. Introduction

This studyis focused on proving the existence of common fixed points of f-contraction mapping defined on complete metric spaces endowed with a partial order by using generalized altering distance functions. I tried to answer the questions how can we prove the existence of common fixed points of f-contraction mappings defined on complete metric spaces endowed with a partial order by using generalized altering distance functions?

Yan *et al.* (2012) established a new contraction mapping principle in partially ordered metric spaces. Su (2014) has proved some fixed point theorems of generalized contraction mappings in a complete metric space endowed with a partial order by using generalized altering distance functions. In recent years, many results appeared related to fixed point theorem in complete metric spaces endowed with a partial ordering (Amini-Harandi and Emami, 2010; Ciri *et al.,* 2008; Naidu, 2003; Suzuki, 2008; Yan *et al.,* 2012).I inspired and motivated by the results mentioned on (Yan *et al.,* 2012) and (Su, 2012), I extend the main theorem of (Su, 2012) to f-contraction mapping in a complete metric space endowed with a partial order by using generalized altering distance functionswith examples. In (Arvanitakis, 2003; Amini-Harandi and Emami, 2010; Babu *et al.,* 2007; Beg *et al.,* 2006; Boyd and Wong, 1969; Chidume *et al.,* 2007; Choudhury *et al.,* 2000), the authors proved some types of weak contractions in complete metric spaces. In particular the existence of a fixed point for weak contraction is extended to partial ordered metric spaces in (Amini-Harandi and Emami, 2010; Choudhury *et al.,* 2000; Harjani and Sadarangni, 2009).

2. Basic Facts and Definitions

Definition 2.1. (Khan *et al.,* 1984) A function $\eta:[0,\infty)\to[0,\infty)$ is called an altering distance function if the following properties are satisfied:

a.$\eta$is continuous and monotonically non-decreasing.
b.$\eta(t)=0$if and only if $t=0$.

Example 2.1.1. The following function is an altering distance function

$$\eta(t)=\begin{cases}0, & t=0\\ at, & t\geq 1,\end{cases} \text{where} a\geq 1.$$

Theorem 2.1. (Khan *et al.,* 1984) Let (X,d)be a complete metric space, let η be an altering distance function, and let $f:X\to X$ be a self-mapping which satisfies the following inequality:

$$\eta(d(fx,fy))\leq c\eta(d(x,y))$$

for all $x, y \in X$and for some $0 \le c < 1$. Then $f$has a unique fixed point.

Definition 2.2. (Ciri *et al.,* 2008) We shall say that the mapping S is f-non-decreasing (resp. f-non-increasing) if $fx \preccurlyeq fy \Rightarrow Sx \preccurlyeq Sy$ (respectively $fx \preccurlyeq fy \Rightarrow Sy \preccurlyeq Sx$) holds for each $x, y \in X$.

Definition 2.3. Consider a function $S: \mathbb{R} \to \mathbb{R}$ and a point $x_0 \in \mathbb{R}$. The function S is said to be upper (resp. lower) semi-continuous at the point x_0 if

$S(x_0) \ge \lim_{x \to x_0} \sup S(x)$, (resp. $S(x_0) \le \lim_{x \to x_0} \inf S(x)$).

Theorem 2.2. (Su, 2014) Let$(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric din X such that $(X, d)$is a complete metric space. Let $T : X \to X$ be a continuous and non-decreasing mapping such that

$$\eta\big(d(Tx, Ty)\big) \le \varphi\big(d(x, y)\big), \forall y \preccurlyeq x,$$

where $\eta$is a generalized altering distance function and$\varphi: [0, \infty) \to [0, \infty)$is a right uppersemi-continuous function with the condition: $\eta(t) > \varphi(t)$for all$t > 0$.If there exists$x_0 \in X$such that $x_0 \preccurlyeq Tx_0$,then T has a fixed point.

Definition 2.4 Let $f$and Sbe self maps of a metric space (X, d). The pair (f, S) is called occasionally weakly compatible (OWC) if there exists $x \in X$which is a coincidence point for f and S at which f and S commute (i.e. if $f(S(x)) = S(f(x))$ for some $x \in C(f, S)$).

3. Main result

Definition 3.1. Let (X, d)be a metric space and $S, f: X \to X$ be two self-maps. A mapping S is said to be f-contraction with generalized altering distance function if there exist $\eta \in \mathrm{H}$ and $\varphi \in \Phi$ such that

$$\eta(d(Sx, Sy)) \le \varphi(d(fx, fy))\text{for all } x, y \in X.$$

Definition 3.2. A point $y \in X$is called point of coincidence of two mappings $f, S: X \to X$if there exists a point $x \in X$ such that $y = fx = Sx$. In this case x is called the coincidence point of f and S and the set of coincidence points of f and S is denoted by $C(f, S)$.

Definition 3.3. Let (X, d) be a metric space and f, S be two self-mappings on (X, d). A point $z \in X$is said to be a common fixed point of $f$and Sif $fz = Sz = z$.

Theorem 3.1. Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric $d$on Xsuch that (X, d) is a complete metric space. Let $f, T: X \to X$ be two continuous self-maps on X satisfying the following conditions:

i)$TX \subset fX$;
ii)fX is closed;
iii)T is f-non-decreasing;
iv) there exists $x_0 \in X$ such that $fx_0 \preccurlyeq Tx_0$;
v)if $z \in C(f, T)$, then $fz \preccurlyeq f(fz)$.

Such that

$$\eta(d(Tx, Ty)) \le \varphi(d(fx, fy)),\ \forall x, y \in X \text{ with, } fy \preccurlyeq fx \quad (1)$$

whereη is a generalized altering distance function and $\varphi: [0, \infty) \to [0, \infty)$ is a right upper semi-continuous function with the condition$\eta(t) > \varphi(t), \forall t > 0$ and$\varphi(t) = 0 \Leftrightarrow t = 0$. Then f and T have a coincidence point. Furthermore if f and T are occasionally weakly compatible maps, then f and T have common fixed point, in X.

Proof. From condition (iv) we have $x_0 \in X$ such that $fx_0 \preccurlyeq Tx_0$. Since $TX \subset fX$, we can choose $x_1 \in X$ such that $fx_1 = Tx_0$. Again from $TX \subset fX$, we can choose $x_2 \in X$ such that $fx_2 = Tx_1$. Continuing this process, we can choose a sequence $\{y_n\}$ which is called Jungck sequence in X such that

$$fx_{n+1} = Tx_n = y_n, \forall n \ge 0. \quad (2)$$

Since $fx_0 \preccurlyeq Tx_0$ and $fx_1 = Tx_0$, we have $fx_0 \preccurlyeq fx_1$. Then by (iii), we have

$$Tx_0 \preccurlyeq Tx_1. \quad (3)$$

Thus by (2) we obtain$fx_1 \preccurlyeq fx_2$. Again by (iii), we have

$$Tx_1 \preccurlyeq Tx_2. \quad (4)$$

That is $fx_2 \preccurlyeq fx_3$. Continuing this process we obtain

$$Tx_0 \preccurlyeq Tx_1 \preccurlyeq Tx_2 \preccurlyeq Tx_3 \preccurlyeq \cdots \preccurlyeq Tx_n \preccurlyeq Tx_{n+1} \preccurlyeq \cdots. \quad (5)$$

Now considering (2) (i.e. $y_n = Tx_n = fx_{n+1}$), from (5) we note that y_n and y_{n+1} are comparable $n \ge 0$.

Case (i) Suppose $y_{n_0} = y_{n_0+1}$ for some$n_0 \in \mathbb{N}$.

Since $y_{n_0} = fx_{n_0+1} = Tx_{n_0}$and $y_{n_0+1} = Tx_{n_0+1}$, we get $fx_{n_0+1} = Tx_{n_0+1}$. This implies x_{n_0+1} is a coincidence point of f and T and hence $x_{n_0+1} \in C(f, T)$ so that $C(f, T) \ne \emptyset$.

Since f and T are occasionally weakly compatible, there exist $p \in C(f, T)$ such that $fTp = Tfp$. Now let $q = fp = Tp$. Then we have $fq = Tq$.

Next we show that $Tq = fq = q$.

Suppose that $Tq \ne q$.Then by condition (v), we have $fp \preccurlyeq f(fp) = fq$ and hence using the contraction condition (1) we obtain

$$\eta\big(d(Tq, q)\big) = \eta\big(d(Tq, Tp)\big) \le \varphi\big(d(fq, fp)\big) = \varphi\big(d(Tq, q)\big) < \eta\big(d(Tq, q)\big),$$

which implies
$\eta\big(d(Tq, q)\big) < \eta(d(Tq, q))$,
a contradiction, since $d(Tq, q) > 0$. Thus, $q = Tq = fq$.

Case (ii) Suppose that $y_n \ne y_{n+1}, \forall n \in \mathbb{N}$.

Now from the contractive condition (1), we obtain

$$\eta\big(d(y_{n+1}, y_n)\big) = \eta\big(d(Tx_{n+1}, Tx_n)\big) \le \varphi\big(d(fx_{n+1}, fx_n)\big) = \varphi(d(y_n, y_{n-1})) < \eta(d(y_n, y_{n-1})).$$

This implies that

$$\eta(d(y_{n+1}, y_n)) < \eta(d(y_n, y_{n-1})) \quad (6)$$

By the non-decreasingness of η, from (6) we get

$$d(y_{n+1}, y_n) < d(y_n, y_{n-1}) \quad (7)$$

Hence, the sequence $\{d(y_n, y_{n+1})\}$ is a decreasing sequence and consequently there exists $r \geq 0$ such that

$d(y_{n+1}, y_n) \to r$, as $n \to \infty$

Now we claim that $r = 0$. Suppose $r > 0$.

$$\eta(d(Tx_{n+1}, Tx_n)) \leq \varphi(d(fx_{n+1}, fx_n)) \quad (8)$$

Considering the non-decreasingness of η and the upper semi-continuity of φ, and letting $n \to \infty$ in (8) we get

$$\eta(r) \leq \lim_{n\to\infty} \sup \eta(d(y_{n+1}, y_n)) \leq \lim_{n\to\infty} sup\varphi(d(y_n, y_{n-1})) \leq \varphi(r).$$

Hence, we have
$\eta(r) \leq \varphi(r)$.

Consequently, we obtain
$\eta(r) < \eta(r)$,

which is impossible since $r > 0$. Thus $r = 0$. Hence
$$d(y_{n+1}, y_n) \to 0 \quad (9)$$

Here we claim that $\{y_n\}$ is a Cauchy sequence.

Now, suppose that $\{y_n\}$ is not a Cauchy sequence. Then there exists a positive real number ε such that for a given $N \in \mathbb{N}$ there exists $m, n \in \mathbb{N}$ such that $m > n > N$ and $d(y_m, y_n) \geq \varepsilon$. Since $\{d(y_{n+1}, y_n)\}$ converges to zero, it follows that there exist strictly increasing sequences $\{n_k\}$ and $\{m_k\}$, $k \geq 1$ of positive integers such that $1 < n_k < m_k$,

$$d(y_{m_k}, y_{n_k}) \geq \varepsilon, \quad \forall k \geq 1 \quad (10)$$

and

$$d(y_{m_k-1}, y_{n_k}) < \varepsilon \quad (11)$$

Using the triangular inequality and the conditions (10) and (11) we have

$$\varepsilon \leq d(y_{m_k}, y_{n_k}) \leq d(y_{m_k}, y_{m_k-1}) + d(y_{m_k-1}, y_{n_k}) < d(y_{m_k}, y_{m_k-1}) + \varepsilon$$

Letting $k \to \infty$ and using (7), we obtain

$$\lim_{k\to\infty} d(y_{m_k}, y_{n_k}) = \varepsilon \quad (12)$$

Using the triangular inequality, we obtain

$$d(y_{m_k-1}, y_{n_k-1}) \leq d(y_{m_k-1}, y_{m_k}) + d(y_{m_k}, y_{n_k}) + d(y_{n_k}, y_{n_k-1}),$$

and

$$d(y_{m_k}, y_{n_k}) \leq d(y_{m_k}, y_{m_k-1}) + d(y_{m_k-1}, y_{n_k-1}) + d(y_{n_k-1}, y_{n_k}).$$

Now letting $k \to \infty$ in the above two inequalities and using (12), we have

$$\lim_{k\to\infty} d(y_{m_k-1}, y_{n_k-1}) = \varepsilon \quad (13)$$

Since η is non-decreasing on $[0, \infty)$, from (10) we have,

$$\eta(\varepsilon) \leq \eta\left(d(y_{n_k}, y_{m_k})\right), \forall k \geq 1, \quad (14)$$

As $m_k > n_k$, by (5), $y_{m_{k-1}}$ and $y_{n_{k-1}}$ are comparable. So from the condition (1), using (5) and the upper semi-continuity of φ, we have

$$\eta(\varepsilon) \leq \limsup_{k\to\infty} \eta\left(d(y_{m_k}, y_{n_k})\right) = \limsup_{k\to\infty} \eta\left(d(Tx_{m_k}, Tx_{n_k})\right)$$

$$\leq \limsup_{k\to\infty} \varphi\left(d(y_{m_k-1}, y_{n_k-1})\right) \leq \varphi(\varepsilon).$$

This implies
$\eta(\varepsilon) \leq \varphi(\varepsilon) < \eta(\varepsilon)$,

which is impossible since $\varepsilon > 0$.

Thus the sequence $\{y_n\}$ is a Cauchy sequence in X.

Since (X, d) is a complete metric space, there exists $y \in X$ such that $y_n \to y$ as $n \to \infty$.

By (2), $\{y_n\} \subseteq fX$ where $y_n = fx_{n+1}$, for each $n = 1,2,3,\cdots$ and fX is closed then there exists $p \in X$ such that $y = fp$.

Next we show that $Tp = y$.

Now by the continuity of f and T, we obtain

$$\eta(d(Tp, y)) = \eta\left(d\left(Tp, \lim_{n\to\infty} Tx_n\right)\right)$$

$$= \eta\left(d\left(Tp, T(\lim_{n\to\infty} x_n)\right)\right)$$

$$\leq \varphi\left(d\left(fp, f(\lim_{n\to\infty} x_n)\right)\right)$$

$$= \varphi\left(d\left(fp, \lim_{n\to\infty} fx_n\right)\right)$$

$$= \varphi(d(fp, fp)) = 0.$$

This implies that $\eta(d(Tp, y)) = 0$ and hence $d(Tp, y) = 0$. As a result we have

$$Tp = y = fp \quad (15)$$

Thus p is a coincidence point of f and T, which implies $C(f, T) \neq \emptyset$. Since f and T are occasionally weakly compatible pair of self maps, f and T commute at some $z \in C(f, T)$.

Now set $w = fz = Tz$. Since f and T are occasionally weakly compatible,

$fw = f(Tz) = T(fz) = Tw$,
which implies

$$fw = Tw \quad (16)$$

Next we claim that $fw = Tw = w$. Suppose $Tw \neq w$. By the condition (v), we have

$$fz \preccurlyeq f(fz) = fw.$$

Then

$$\eta(d(Tw, w)) = \eta(d(Tw, Tz)) \leq \varphi(d(fw, fz)) = \varphi(d(Tw, w)) < \eta(d(Tw, w))$$

which implies that

$$\eta(d(Tw, w)) < \eta(d(Tw, w)),$$

a contradiction. Thus $Tw = w$. And hence by (15), we have

$$fw = Tw = w.$$

Thus, we have proved that f and T have a common fixed pointin X.

The following is an example in support of Theorem 3.1.

Example 3.1.1. Let $X = \{-2, -1, 0, 1\}$. We define a partial order "$\preccurlyeq$" on X by

$$\preccurlyeq = \{(-2,-2), (-1,-1), (0,0), (1,1), (0,-1), (1,0), (1,-1)\}.$$

Let d be the usual metric on X. Define $f, T: X \to X$by

$f(-1) = 1, f(0) = 0, f(1) = -2, f(-2) = -1$, and
$T(-1) = 0, T(0) = 0, T(1) = 1, T(-2) = 0$.

Then $T(X) = \{0,1\}$ and $f(X) = \{-2, -1, 0, 1\}$ and hence $T(X) \subset f(X)$ and $f(X) = \{-2, -1, 0, 1\}$ is closed.

Next we show that T is f-non-decreasing.

$-2 = f(1) \preccurlyeq f(1) = -2 \Rightarrow 1 = T(1) \preccurlyeq T(1) = 1$;
$-1 = f(-2) \preccurlyeq f(-2) = -1 \Rightarrow 0 = T(-2) \preccurlyeq T(-2) = 0$;
$0 = f(0) \preccurlyeq f(0) = 0 \Rightarrow 0 = T(0) \preccurlyeq T(0) = 0$;
$1 = f(-1) \preccurlyeq f(-1) = 1 \Rightarrow 0 = T(-1) \preccurlyeq T(-1) = 0$;
$0 = f(0) \preccurlyeq f(-2) = -1 \Rightarrow 0 = T(0) \preccurlyeq T(-2) = 0$;
$1 = f(-1) \preccurlyeq f(0) = 0 \Rightarrow 0 = T(-1) \preccurlyeq T(0) = 0$; and
$1 = f(-1) \preccurlyeq f(-2) = -1 \Rightarrow 0 = T(-1) \preccurlyeq T(-2) = 0$.

This shows that T is f-non-decreasing. We also observe that $f(1) \preccurlyeq T(1)$ and $z = 0 \in C(f, T)$ such that $fz \preccurlyeq ffz$.

Now we show that f and T satisfy the contraction condition of Theorem 3.1 with $\eta(t) = \frac{1}{2}t$ and $\varphi(t) = \frac{1}{3}t$.

$$\eta\left(d(T(0), T(-2))\right) = 0 \leq \frac{1}{3} = \varphi(d(0, -1)) = \varphi\left(d(f(0), f(-2))\right);$$

$\eta\left(d(T(-1), T(0))\right) = 0 \leq \frac{1}{3} = \varphi(d(1,0)) = \varphi\left(d(f(-1), f(0))\right)$;and $\eta\left(d(T(-1), T(-2))\right) = 0 \leq \frac{2}{3} = \varphi(d(1,-1)) = \varphi\left(d(f(-1), f(-2))\right)$.

Thus, the pair of mappings f and T satisfy all conditions of Theorem 3.1 and 0 is the common fixed point of f and T.

Remark 1: If we choose $f = I_X$=The identity map on X, Theorem 2.1 follows as corollary to Theorem 3.1.

Note that the map T in Example 3.1 is a non-decreasing map, since

$-2 \preccurlyeq -2 \Rightarrow 0 = T(-2) \preccurlyeq T(-2) = 0$;
$-1 \preccurlyeq -1 \Rightarrow 0 = T(-1) \preccurlyeq T(-1) = 0$;
$0 \preccurlyeq 0 \Rightarrow 0 = T(0) \preccurlyeq T(0) = 0$;
$1 \preccurlyeq 1 \Rightarrow 1 = T(1) \preccurlyeq T(1) = 1$;
$0 \preccurlyeq -1 \Rightarrow 0 = T(0) \preccurlyeq T(-1) = 0$;
$1 \preccurlyeq 0 \Rightarrow 1 = T(1) \preccurlyeq T(0) = 0$;and
$1 \preccurlyeq -1 \Rightarrow 1 = T(1) \preccurlyeq T(-1) = 0$.

So in Example 3.1, if we choose $f = I_X$=The identity map on X, one can observe that for $x = 1$ and $y = 0$, where $(1,0) \in \preccurlyeq$, we get $\eta(1) \leq \varphi(1)$ which absurd and hence the selfmap T cannot satisfy the contraction condition of Su [26] for any φ and η such that $\eta(t) > \varphi(t) \forall t > 0$.

4. Conclusion

In this work I developed f-contraction mappings and obtained a new common fixed point theorem for f-contraction mapping in a complete metric space endowed with a partial order by using generalized altering distance functions and common fixed point theorems obtained are proved. This theorem will help us to develop many theorems in further development of f-contraction mappings.

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