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RESEARCH ARTICLE

ON TWO IMPORTANT CLASSES OF (α, β)-METRICS BEING PROJECTIVELY RELATED

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ABSTRACT

The treatment of choice for a In this article, we find the necessary and sufficient condition under which the (α, β)-metric $F = c_1\alpha + c_2\beta + \frac{\beta^2}{\alpha}$; $c_2 \neq 0$ is projectively related to a Kropina metric on a manifold M of dimension $n \geq 3$, where α and $\bar{\alpha}$ are two Riemannian metrics, β and $\bar{\beta}$ are two non zero 1-forms.

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INTRODUCTION

Two regular metrics are called projectively related if there is a diffeomorphism between them such that the pull back metric is point wise projective to another one. In Riemannian geometry, two Riemannian metrics α and $\bar{\alpha}$ on a manifold M are projectively related if and only if their spray coefficients have the relation $G^i_\alpha = \bar{G}^i_{\bar{\alpha}} + P_0 y^i$, where $P = P(x)$ is a scalar function on M and $P_0 = P_{x^k} y^k$. In Finsler geometry, two Finsler metrics F and $\bar{F}$ on a manifold M are called projectively related if $G^i = \bar{G}^i + Py^i$, where G^i *and* $\bar{G}^i$ are the geodesic coefficients of F and $\bar{F}$, respectively and $P = P(x, y)$ is a scalar function on the tangent bundle TM_0. In this case, any geodesic of the first is also geodesic for the second and vice versa. The projective change between two Finsler spaces have been studied by (Bacso, 1994; Feng Mu and Xinyue Cheng, 2012; Narasimhamurthy, 2014; Narasimhamurthy, 2012; Tayebi, 2013; Zohrehvand, 2011). In order to find explicit examples of projectively related Finsler metrics, we consider (α, β)-metrics. (α, β)-metrics form a special and very important class of Finsler metrics which can be expressed in the form $= \alpha\varphi(s)$; $s = \frac{\beta}{\alpha}$, where α is a Riemannian metric and β is a 1-form and φ is a C^∞ positive function on the definite domain. In particular, when $= \frac{1}{s}$, the Finsler metric $F = \frac{\alpha^2}{\beta}$ is called Kropina metric. Kropina metric was first introduced by L. Berwald in connection with two dimensional Finsler space with rectilinear extremal and was investigated by V. K. Kropina (Kropina, 1952). They together with Randers metric are C-reducible (Matsumto, 1978). However, Randers metric are regular Finsler metrics but Kropina metric are non-regular Finsler metrics. Kropina metric seem to be among the simplest nontrivial Finsler metric with many interesting applications in physics, electron optics with a magnetic field, dissipative mechanics and irreversible thermodynamics (Ingarden, 1987; Shibat, 1978).

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Also, they have interesting applications in relativistic field theory, evolution and developmental biology. By (Feng Mu, 2012), for the projective equivalence between a general (α,β)-metric and a Kropina metric, we have the following lemma:

Lemma 1.1: Let $F=\alpha\varphi\left(\frac{\beta}{\alpha}\right)$ be an (α,β)-metric on n-dimensional manifold $M\ (n\geq 3)$ satisfying that β is not parallel with respect to $\alpha, db\neq 0$ everywhere (or) b = constant and F is not of Randers type. Let $\bar{F}=\frac{\bar{\alpha}^2}{\bar{\beta}}$ be a Kropina metric on the manifold M, where $\bar{\alpha}=\lambda(x)\alpha$ and $\bar{\beta}=\mu(x)\beta$. Then F is projectively equivalent to $\bar{F}$ if and only if the following equations holds

$$[1+(k_1+k_2s^2)s^2+k_3s^2]\varphi''=(k_1+k_2s^2)(\varphi-s\varphi'),$$

$$G^i_\alpha=\bar{G}^i_{\bar{\alpha}}+\theta y^i-\sigma(k_1\alpha^2+k_2\beta^2)b^i,$$

$$b_{i|j}=2\sigma[(1+k_1b^2)a_{ij}+(k_2b^2+k_3)b_ib_j],$$

$$\bar{s}_{ij}=\frac{1}{\bar{b}^2}(\bar{b}_i\bar{s}_j-\bar{b}_j\bar{s}_i)$$

where $\sigma=\sigma(x)$ is a scalar function and k_1, k_2 and k_3 are constants. In this case, both $F=\alpha\varphi\left(\frac{\beta}{\alpha}\right)$ and $\bar{F}=\frac{\bar{\alpha}^2}{\bar{\beta}}$ are Douglas metrics. The purpose of this paper is to study the projective relation between an (α,β)-metric $F=c_1\alpha+c_2\beta+\frac{\beta^2}{\alpha};\ c_2\neq 0$ and Kropina metric $\bar{F}=\frac{\bar{\alpha}^2}{\bar{\beta}}$. The main results of the paper are as follows.

Theorem 1.1: Let $F=c_1\alpha+c_2\beta+\frac{\beta^2}{\alpha};\ c_2\neq 0$ be an (α,β)- metric and $\bar{F}=\frac{\bar{\alpha}^2}{\bar{\beta}}$ be a Kropina metric on an n-dimensional manifold $M\ (n\geq 3)$ where α and $\bar{\alpha}$ are two Riemannian metrics, β and $\bar{\beta}$ are two non zero collinear 1-forms. Then F is projectively equivalent to $\bar{F}$ if and only if they are Douglas metrics and the geodesic coefficients of α and $\bar{\alpha}$ have the following relation,

$$G^i_\alpha+2\alpha^2\tau b^i=\bar{G}^i_{\bar{\alpha}}+\frac{1}{2\bar{b}^2}\left(\bar{\alpha}^2\bar{s}^i+\bar{r}_{00}\bar{b}^i\right)+\theta y^i,$$

where $b^i=a^{ij}b_j, \bar{b}^i=\bar{a}^{ij}\bar{b}_j,\ \bar{b}^2=\|\bar{\beta}\|^2_{\bar{\alpha}}$ and $\tau=\tau(x)$ is scalar function and $\theta=\theta_i y^i$ is a 1-form on M.

By (Li, 2009) and (9), we obtain immediately from theorem 1.1, that

Proposition 1: Let $F=c_1\alpha+c_2\beta+\frac{\beta^2}{\alpha};\ c_2\neq 0$ be an (α,β)-metric and $\bar{F}=\frac{\bar{\alpha}^2}{\bar{\beta}}$ be a Kropina metric on an n-dimensional manifold $M\ (n\geq 3)$ where α and $\bar{\alpha}$ are two Riemannian metrics, β and $\bar{\beta}$ are two non zero collinear 1-forms. Then F is projectively equivalent to $\bar{F}$ if and only if the following equations holds,

$$G^i_\alpha+2\alpha^2\tau b^i=\bar{G}^i_{\bar{\alpha}}+\frac{1}{2\bar{b}^2}\left(\bar{\alpha}^2\bar{s}^i+\bar{r}_{00}\bar{b}^i\right)+\theta y^i,\qquad b_{i|j}=2\tau\left\{\left(1+\frac{2b^2}{c_1}\right)a_{ij}-\left(\frac{3}{c_1}\right)b_ib_j\right\}$$

$$\bar{s}_{ij}=\frac{1}{\bar{b}^2}(\bar{b}_i\bar{s}_j-\bar{b}_j\bar{s}_i)$$

Where $b_{i|j}$ denote the coefficient of the covariant derivatives of β with respect to α.

PRELIMINARIES

The terminology and notations are referred to (1), (7), (12). Let $F_n=(M,F)$ be a Finsler space on a differential manifold M endowed with a fundamental function $F(x,y)$. We use the following notations:

- $g_{ij}=\frac{1}{2}\dot{\partial}_i\dot{\partial}_jF^2,\ \dot{\partial}_i=\frac{\partial}{\partial y^i}$,
- $C_{ijk}=\frac{1}{2}\dot{\partial}_k g_{ij}$,
- $h_{ij}=g_{ij}-l_il_j$

- $\gamma^i_{jk} = \frac{1}{2} g^{ir} (\partial_j g_{rk} + \partial_k g_{rj} - \partial_r g_{jk})$,
- $G^i = \frac{1}{2} \gamma^i_{jk} y^j y^k$, $G^i_j = \dot{\partial}_j G^i$, $G^i_{jk} = \dot{\partial}_k G^i_j$, $G^i_{jkl} = \dot{\partial}_l G^i_{jk}$.

The concept of (α, β)-metric $L(\alpha, \beta)$ was introduced in 1972 by M. Matsumoto and studied by many others. The Finsler space $F^n = (M, F)$ is said to have an (α, β)-metric if F is positively homogeneous function of degree one in two variables $\alpha^2 = a_{ij}(x) y^i y^j$ and $\beta = b_i(x) y^i$. A change $F \to \bar{F}$ of a Finsler metric on a same underlying manifold M is called projective change if any geodesic in (M, F) remains to be a geodesic in $(M, \bar{F})$ and vice versa. We say that a Finsler metric is projectively related to another Finsler metric if they have the same geodesic as point sets. In Riemannian geometry, two Riemannian metrics α and $\bar{\alpha}$ are projectively related if and only if their spray coefficients have the relation (Narasimhamurthy, 2014),

$$G^i_\alpha = G^i_{\bar{\alpha}} + \lambda_{x^k} y^k y^i, \tag{2.1}$$

where $\lambda = \lambda(x)$ is a scalar function on the based manifold and (x^i, y^j) denotes the local coordinates in the tangent bundle TM Two Finsler metrics F and $\bar{F}$ on a manifold M are called projectively related if and only if their spray coefficients have the relation (Narasimhamurthy, 2014),

$$G^i = \bar{G}^i + P(y) y^i \tag{2.2}$$

where $P(y)$ is a scalar function on $TM \backslash \{0\}$ and homogeneous of degree one in y. For a given Finsler metric $F = F(x, y)$, the geodesic of F satisfy the following ODE:

$$\frac{d^2 x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0,$$

where $G^i = G^i(x, y)$ is called the geodesic coefficient, which is given by

$$G^i = \frac{1}{4} g^{il} \{[F^2]_{x^m y^l} y^m - [F^2]_{x^l}\}.$$

Let $\varphi = \varphi(s)$, $|s| < b_0$, be a positive C^∞ function satisfying the following

$$\varphi(s) - s\varphi'(s) + (b2 - s2)\varphi''(s) > 0 \qquad (|s| \le b < b_0). \tag{2.3}$$

If $\alpha = \sqrt{a_{ij} y^i y^j}$ is a Riemannian metric and $\beta = b_i y^i$ is 1-form satisfying $\|\beta_x\|_\alpha < b_0 \quad \forall x \in M$, then $F = \alpha\varphi(s)$, $s = \frac{\beta}{\alpha}$, is called a regular (α, β)-metric. In this case, the fundamental form of the metric tensor induced by F is positive definite. Let $\nabla\beta = b_{i|j} dx^i \otimes dx^j$ be covariant derivative of β with respect to α.

Denote $r_{ij} = \frac{1}{2}(b_{i|j} + b_{j|i})$ and $s_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i})$.

Note that β is closed if and only if $s_{ij} = 0$ (Shen, 2004). Let $s_j = b^i s_{ij}$, $s^i_j = a^{il} s_{lj}$, $s_0 = s_i y^i$, $s^i_0 = s^i_j y^j$ and $r_{00} = r_{ij} y^i y^j$. The relation between the geodesic coefficients G^i of F and geodesic coefficients G^i_α of α is given by

$$G^i = G^i_\alpha + \alpha Q s^i_0 + \{-2Q\alpha s_0 + r_{00}\}\{\Psi b^i + \theta\alpha^{-1} y^i\}, \tag{24}$$

where

$$\theta = \frac{\varphi\varphi' - s(\varphi\varphi'' + \varphi'\varphi')}{2\varphi\{(\varphi - s\varphi') + (b^2 - s^2)\varphi''\}},$$

$$Q = \frac{\varphi'}{\varphi - S\varphi'},$$

$$\Psi = \frac{1}{2} \frac{\varphi''}{\{(\varphi - S\varphi') + (b^2 - s^2)\varphi''\}}.$$

For a Kropina metric $F = \frac{\alpha^2}{\beta}$, it is very easy to see that it is not a regular (α,β)-metric but the relation $\varphi(s) - s\varphi'(s) + (b^2 - s^2)\varphi''(s) > 0$ is still true for $|s| > 0$.

In (Li et al., 2009), the authors characterized the (α,β)-metrics of Douglas type.

Lemma 2.2: Let $F = \alpha\varphi\left(\frac{\beta}{\alpha}\right)$ be a regular (α,β)-metric on an n-dimensional manifold $M \quad (n \geq 3)$. Assume that β is not parallel with respect to α and $db \neq 0$ everywhere or b= constant, and F is not of Randers type. Then F is a Douglas metric if and only if the function $\varphi = \varphi(s)$ with $\varphi(0) = 1$ satisfies the following

$$[1 + (k_1 + k_2 s^2)s^2 + k_3 s^2]\varphi'' = (k_1 + k_2 s^2)(\varphi - s\varphi'). \tag{2.5}$$

and β satisfies

$$b_{i|j} = 2\sigma\left[(1 + k_1 b^2)a_{ij} + (k_2 b^2 + k_3)b_i b_j\right] \tag{2.6}$$

where $b^2 = \|\beta\|_\alpha^2$ and $\sigma = \sigma(x)$ is a scalar function and k_1, k_2, k_3 are constants with $(k_1, k_2) \neq (0,0)$.
For a Kropina metric, we have the following

Lemma 2.3.(9): Let $F = \frac{\alpha^2}{\beta}$ be Kropina metric on an n-dimensional manifold M. Then (i) $(n \geq 3)$ Kropina metric F with $b^2 \neq 0$ is Douglas metric if and only if

$$s_{ik} = \frac{1}{b^2}(b_i s_k - b_j s_i). \tag{2.7}$$

(ii) $(n = 2)$ Kropina metric F is a Douglas metric. Definition 2.1. (10): Let

$$D^i_{jkl} = \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(G^i - \frac{1}{n+1}\frac{\partial G^m}{\partial y^m}y^i\right) \tag{2.8}$$

where G^i is the spray coefficients of F. The tensor $D = D^i_{jkl}\partial_i \otimes dx^j \otimes dx^k \otimes dx^l$ is called the Douglas tensor. A Finsler metric is called Douglas metric if the Douglas tensor vanishes. We know that the Douglas tensor is a projective invariant. Note that the spray coefficients of a Riemannian metric are quadratic forms and one can see that the Douglas tensor vanishes from (2.8). This shows that Douglas tensor is a non-Riemannian quantity. In the following, we use quantities with a bar to denote the corresponding quantities of the metric $\bar{F}$. Now, first we compute the Douglas tensor of a general (α,β)-metric.

Let

$$\hat{G}^i = G^i_\alpha + \alpha Q s^i_0 + \Psi\{-2Q\alpha s_0 + r_{00}\}b^i. \tag{2.9}$$

then (2.4) becomes

$$G^i = \hat{G}^i + \theta\{-2Q\alpha s_0 + r_{00}\}\alpha^{-1}y^i.$$

Clearly, G^i and $\hat{G}^i$ are projective equivalent according to (2.2), they have the same Douglas tensor.

Let

$$T^i = \alpha Q s^i_0 + \Psi\{-2Q\alpha s_0 + r_{00}\}b^i. \tag{2.10}$$

Then $\hat{G}^i = G^i_\alpha + T^i$, thus $D^i_{jkl} = \hat{D}^i_{jkl}$.

$= \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(G_\alpha^i - \frac{1}{n+1}\frac{\partial G_\alpha^m}{\partial y^m} y^i + T^i - \frac{1}{n+1}\frac{\partial T^m}{\partial y^m} y^i\right)$

$$\frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(T^i - \frac{1}{n+1}\frac{\partial T^m}{\partial y^m} y^i\right) \qquad (2.11)$$

To compute (2.11) explicitly, we use the following identities

$\alpha_{y^k} = \alpha^{-1} y_k; \; s_{y^k} = \alpha^{-2}(b_k \alpha - s y_k),$

where $y_i = a_{il} y^l$.

Hereafter, α_{y^k} means $\frac{\partial \alpha}{\partial y^k}$. Then

$[\alpha Q s_0^m]_{y^m} = \alpha^{-1} y_m Q s_0^m + \alpha^{-2} Q' [b_m \alpha^2 - \beta y_m] s_0^m = Q' s_0$

and

$[\Psi(-2Q\alpha s_0 + r_{00}) b^m]_{y^m} = \Psi' \alpha^{-1}(b^2 - s^2)[r_{00} - 2Q\alpha S_0] + 2\Psi[r_0 - Q'(b^2 - s^2)s_0 - Qss_0]$

where $r_j = b^i r_{ij}$ *and* $r_0 = r_j y^j$. Thus from (2.10), we have

$$T^m_{y^m} = Q' s_0 + \Psi' \alpha^{-1}(b^2 - s^2)[r_{00} - 2Q\,\alpha s_0] + 2\Psi[r_0 - Q'(b^2 - s^2)s_0 - Qss_0] \qquad (2.12)$$

Let F and $\bar{F}$ be two (α, β)-metrics, we assume that they have the same Douglas tensor, i.e.

$D^i_{jkl} = \bar{D}^i_{jkl}.$

From (2.8) and (2.11), we have

$\frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(T^i - \bar{T}^i - \frac{1}{n+1}(T^m_{y^m} - \bar{T}^m_{y^m}) y^i\right) = 0$

Then there exists a class of scalar function $H^i_{jk} = H^i_{jk}(x)$, such that

$$H^i_{00} = T^i - \bar{T}^i - \frac{1}{n+1}(T^m_{y^m} - \bar{T}^m_{y^m}) y^i, \qquad (2.13)$$

where $H^i_{00} = H^i_{jk} y^j y^k$, T^i *and* $T^m_{y^m}$ are given by (2.10) and (2.12) respectively.

Projective relation between two important classes of (α, β)-Metrics

In this section, we find the projective relation between special metric (α, β)-metric $F = c_1\alpha + c_2\beta + \frac{\beta^2}{\alpha}; \; c_2 \neq 0$ and $\bar{F} = \frac{\alpha^2}{\beta}$ on a same underlying manifold M of dimension $n \geq 3$

For an (α, β)-metric $F = c_1\alpha + c_2\beta + \frac{\beta^2}{\alpha}; \; c_2 \neq 0,$ one can prove (2.3) that F is a regular Finsler metric if and only if 1-form β satisfies the condition $\|\beta_x\|_\alpha < 1$ for any $x \in M$.

The geodesic coefficients are given by (2.4) with

$\theta = \frac{(c_1 c_2 - c_2 s^2 - 4s^3)}{2(c_1 + c_2 s + s^2)(c_1 + 2b^2 - 3s^2)},$

$Q = \frac{c_2 + 2s}{c_1 - s^2},$

$$\Psi = \frac{1}{(c_1+2b^2-3s^2)}, \tag{3.1}$$

For Kropina metric $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$, the geodesic coefficient are given by (2.4) with

$$\bar{Q} = -\frac{1}{2s}$$
$$\bar{\theta} = -\frac{s}{\bar{b}^2}$$
$$\bar{\Psi} = \frac{1}{2\bar{b}^2}. \tag{3.2}$$

Now, we have the following theorem

Theorem 3.2: Let $F = c_1\alpha + c_2\beta + \frac{\beta^2}{\alpha};\ c_2 \neq 0$ be a special (α,β)- metric and $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ be a Kropina metric on an n-dimensional manifold $M\ (n \geq 3)$ where α and $\bar{\alpha}$ are two Riemannian metrics, β and $\bar{\beta}$ are two non zero 1-forms. Then F and $\bar{F}$ have the same Douglas tensors if and only if they are Douglas metrics.

Proof: First, we prove the sufficient condition.

Let F and $\bar{F}$ be Douglas metrics and corresponding Douglas tensors be D^i_{jkl} and $\bar{D}^i_{jkl}$. Then by the definition of Douglas metric, we have $D^i_{jkl} = 0$ and $\bar{D}^i_{jkl} = 0$, that is both F and $\bar{F}$ have the same Douglas tensor.

Next, we prove the necessary condition.

If F and $\bar{F}$ have the same Douglas tensor, then (2.13) holds. Plugging (3.1) and (3.2) into (2.13), we have

$$H^i_{00} = \frac{A^i\alpha^9 + B^i\alpha^8 + C^i\alpha^7 + D^i\alpha^6 + E^i\alpha^5 + F^i\alpha^4 + G^i\alpha^3 + H^i\alpha^2 + I^i}{J\alpha^8 + K\alpha^6 + L\alpha^4 + M\alpha^2 + N} + \frac{\bar{A}^i\bar{\alpha}^2 + \bar{B}^i}{2\bar{b}^2\bar{\beta}}, \tag{3.3}$$

where

$$A^i = c_1c_2(c_1+2b^2)\{(c_1+2b^2)s^i_0 - 2s_0b^i\}, \qquad B^i = 2\beta c_1(c_1+2b^2)[\{(c_1+2b^2)s^i_0 - 2s_0b^i\} + c_1r_{00}b^i - 2\lambda y^i c_1(r_0+s_0)]$$
$$C^i = -\beta^2(c_1+2b^2)\{(c_1+2b^2)c_1+6\}c_2s^i_0 + 2\beta^2c_2(4c_1+2b^2)s_0b^i + 12\beta c_1c_2b^2\lambda s_0y^i.$$
$$D^i = -2\beta^3(7c_1+2b^2)s^i_0 + 4\beta^3(4c_1+2b^2)s_0b^i - \beta^2c_1(2c_1c_2+4c_2b^2+3c_1)r_{00}b^i - 6\beta b^2c_1^2r_{00}\lambda y^i +$$
$$2\beta^2c_1(3c_1+4b^2)r_0\lambda y^i + 2\beta^2c_1(5c_1+16b^2)\lambda s_0y^i$$
$$E^i = 3\beta^3[\{3\beta c_1 + 2\beta(c_1+2b^2)\}s^i_0 - 2s_0b^i - 2(c_1+2b^2)s_0\lambda y^i],$$
$$F^i = 6\beta^5\{(5c_1+4b^2)s^i_0 - 2s_0b^i\} + \beta^4c_2\{c_1(c_2+6)+2c_2b^2\}r_{00}b^i + 6\beta^3c_1(c_1+2b^2)r_{00}\lambda y^i - 12\beta^4c_1r_0\lambda y^i$$
$$- 2\beta^4(25c_1+14b^2)$$
$$G^i = 3\beta^5c_2\{-3\beta s^i_0 + 4\lambda s_0y^i\}$$
$$H^i = -18\beta^7s^i_0 - 3\beta^6c_2^2r_{00}b^i - 6\beta^5(2c_1+b^2)r_{00}\lambda y^i - 12\beta^6s_0\lambda y^i, \quad I^i = 6\beta^7r_{00}\lambda y^i,$$

and

$$J = c_1^2(c_1+2b^2)^2, \quad K = -2\beta^2c_1(c_1+2b^2)(4c_1+2b^2), \quad L = \beta^4\{(c_1+2b^2)(2b^2-11c_1)+9c_1^2\}, \quad M = -6\beta^6(4c_1+2b^2),$$
$$N = 9\beta^8$$

and

$$\bar{A}^i = \bar{b}^2\bar{s}^i_0 - \bar{b}^i\bar{s}_0,$$
$$\bar{B}^i = \bar{\beta}[2\lambda y^i(\bar{r}_0+\bar{s}_0) - \bar{b}^i\bar{r}_{00}].$$

Further, (3.3) is equivalent to

$$(A^i\alpha^9 + B^i\alpha^8 + C^i\alpha^7 + D^i\alpha^6 + E^i\alpha^5 + F^i\alpha^4 + G^i\alpha^3 + H^i\alpha^2 + I^i)(2\bar{b}^2\bar{\beta}) + (\bar{A}^i\bar{\alpha}^2 + \bar{B}^i)\times$$

$$(J\alpha^8 + K\alpha^6 + L\alpha^4 + M\alpha^2 + N) = H^i_{00}(2\bar{b}^2\bar{\beta})(J\alpha^8 + K\alpha^6 + L\alpha^4 + M\alpha^2 + N) \quad (3.4)$$

Replacing (y^i) by $(-y^i)$ in (3.4) yields

$$(-A^i\alpha^9 + B^i\alpha^8 - C^i\alpha^7 + D^i\alpha^6 - E^i\alpha^5 + F^i\alpha^4 - G^i\alpha^3 + H^i\alpha^2 + I^i)(-2\bar{b}^2\bar{\beta}) - (\bar{A}^i\bar{\alpha}^2 + \bar{B}^i)$$

$$\times (J\alpha^8 + K\alpha^6 + L\alpha^4 + M\alpha^2 + N) = -H^i_{00}(J\alpha^8 + K\alpha^6 + L\alpha^4 + M\alpha^2 + N)(2\bar{b}^2\bar{\beta}) \quad (3.5)$$

Adding (3.4) and (3.5), we get

$$(A^i\alpha^9 + C^i\alpha^7 + E^i\alpha^5 + G^i\alpha^3)(2\bar{b}^2\bar{\beta}) = 0.$$

Above equation reduces to

$$A^i\alpha^9 + C^i\alpha^7 + E^i\alpha^5 + G^i\alpha^3 = 0. \quad (3.6)$$

Therefore we conclude that (3.3) is equivalent to

$$H^i_{00} = \frac{B^i\alpha^8 + D^i\alpha^6 + F^i\alpha^4 + H^i\alpha^2 + I^i}{J\alpha^8 + K\alpha^6 + L\alpha^4 + M\alpha^2 + N} + \frac{\bar{A}^i\bar{\alpha}^2 + \bar{B}^i}{2\bar{b}^2\bar{\beta}}. \quad (3.7)$$

Equation (3.7) is equivalent to

$$B^i\alpha^8 + D^i\alpha^6 + F^i\alpha^4 + H^i\alpha^2 + I^i)(2\bar{b}^2\bar{\beta}) + \overline{(A}^i\bar{\alpha}^2 + \bar{B}^i) \times$$
$$(J\alpha^8 + K\alpha^6 + L\alpha^4 + M\alpha^2 + N) = H^i_{00}(2\bar{b}^2\bar{\beta})(J\alpha^8 + K\alpha^6 + L\alpha^4 + M\alpha^2 + N) \quad (3.8)$$

From the equation (3.8), we can see that $\bar{A}^i\bar{\alpha}^2(J\alpha^8 + K\alpha^6 + L\alpha^4 + M\alpha^2 + N)$ can be divided by $\bar{\beta}$. Since $\beta = \mu\bar{\beta}$, then $\bar{A}^i\bar{\alpha}^2 J\alpha^8$ can be divided by $\bar{\beta}$. Because $\bar{\beta}$ is prime with respect to α and $\bar{\alpha}$, therefore $\bar{A}^i = \bar{b}^2\bar{s}^i_0 - \bar{b}^i\bar{s}_0$ can be divided by $\bar{\beta}$. Hence there is a scalar function $\Psi^i(x)$ such that

$$\bar{s}^i_0 - \bar{b}^i\bar{s}_0 = \bar{\beta}\Psi^i \quad (3.9)$$

Contracting (3.9) by $\bar{y}_i = \bar{a}_{ij}y^j$, we get $\Psi^i(x) = -\bar{s}^i$. Thus we have

$$\bar{s}_{ij} = \frac{1}{\bar{b}^2}(\bar{b}_i\bar{s}_j - \bar{b}_j\bar{s}_i) \quad (3.10)$$

Thus, by lemma 2.3, $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ is a Douglas metric. i.e. both $F = c_1\alpha + c_2\beta + \frac{\beta^2}{\alpha};\ c_2 \neq 0$

and $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ Douglas metrics.

If $= 2$, $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ is a Douglas metric by lemma 2.3. Thus F and $\bar{F}$ have the same Douglas tensors means that they are Douglas metrics.

Hence the proof.

Now we prove the following main theorem

Theorem 3.3: Let $F = c_1\alpha + c_2\beta + \frac{\beta^2}{\alpha};\ c_2 \neq 0$ be a special $(\alpha, \beta)-$metric and $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ be a Kropina metric on an n-dimensional manifold $M\ (n \geq 3)$ where α and $\bar{\alpha}$ are two Riemannian metrics, β and $\bar{\beta}$ are two non-zero 1-forms. Then F is projectively equivalent to $\bar{F}$ if and only if Douglas metrics and geodesic coefficients of α and $\bar{\alpha}$ have the following relation:

$$G^i_\alpha + 2\alpha^2\tau b^i = G^i_{\bar{\alpha}} + \frac{1}{2\bar{b}^2}(\bar{\alpha}^2\bar{s}^i + \bar{r}_{00}\bar{b}^i) + \theta y^i.$$

where $b^i = a^{ij}b_j$, $\bar{b}^i = \bar{a}^{ij}\bar{b}_j, \bar{b}^2 = \|\bar{\beta}\|^2_{\bar{\alpha}}$ and $\tau = \tau(x)$ is scalar function and $\theta = \theta_i y^i$ is a 1-form on M.

Proof:

First we prove the necessary condition.

If F is projectively related to $\bar{F}$, then they have the same Douglas tensor. By theorem 3.2, we know that F and $\bar{F}$ are Douglas metrics. By (6), we know that $(\alpha,\beta)-$metric $F = c_1\alpha + c_2\beta + \frac{\beta^2}{\alpha};\ c_2 \neq 0$ is a Douglas metric if and only if

$$b_{i|j} = 2\tau\left\{\left(1+\frac{2b^2}{c_1}\right)a_{ij} - \left(\frac{3}{c_1}\right)b_ib_j\right\} \tag{3.11}$$

where $\tau = \tau(x)$ is a scalar function on M. In this case, β is closed. Plugging (3.11) and (3.1) into (2.4) yields

$$G^i = G^i_\alpha + \left\{\frac{\alpha^3c_1c_2 - 3c_2\alpha\beta^2 - 4\beta^3}{c_1\alpha^2 + c_2\alpha\beta + \beta_2}\right\}\tau y^i + 2\tau\alpha^2 b^i \tag{3.12}$$

On the other hand , plugging (3.10) and (3.2) into (2.4), we have

$$\bar{G}^i = \bar{G}^i_{\bar{\alpha}} - \frac{1}{2\bar{b}^2}\left\{-\bar{\alpha}^2\bar{s}^i + \left(2\bar{s}_0y^i - \bar{r}_{00}\bar{b}^i\right) + 2\frac{\bar{r}_{00}\bar{\beta}y^i}{\bar{\alpha}^2}\right\} \tag{3.13}$$

By the projective equivalence of F and $\bar{F}$, then there is a scalar function $P = P(x,y)$ on $TM\backslash\{0\}$ such that

$$G^i = \bar{G}^i + Py^i \qquad (3.14)$$

By (3.12), (3.13) and (3.14), we have

$$\left[P - \left\{\frac{\alpha^3c_1c_2 - 3c_2\alpha\beta^2 - 4\beta^3}{c_1\alpha^2 + c_2\alpha\beta + \beta^2}\right\}\tau - \frac{1}{\bar{b}^2}\left(\bar{s}_0 + \frac{\bar{r}_{00}\bar{\beta}}{\bar{\alpha}^2}\right)\right]y^i = G^i_\alpha - \bar{G}^i_{\bar{\alpha}} + 2\alpha^2\tau b^i - \frac{1}{2\bar{b}^2}\left(\bar{\alpha}^2\bar{s}^i + \bar{r}_{00}\bar{b}^i\right). \quad (3.15)$$

Note that right hand side of (3.15) is quadratic in y. Then there exist a 1-form $\theta = \theta_i y^i$ on M such that $P - \left\{\frac{\alpha^3c_1c_2 - 3c_2\alpha\beta^2 - 4\beta^3}{c_1\alpha^2 + c_2\alpha\beta + \beta^2}\right\}\tau - \frac{1}{\bar{b}^2}\left(\bar{s}_0 + \frac{\bar{r}_{00}\bar{\beta}}{\bar{\alpha}^2}\right) = \theta.$

Thus we have

$$G^i_\alpha + 2\alpha^2\tau b^i = \bar{G}^i_{\bar{\alpha}} + \frac{1}{2\bar{b}^2}\left(\bar{\alpha}^2\bar{s}^i + \bar{r}_{00}\bar{b}^i\right) + \theta y^i. \tag{3.16}$$

This completes the proof of necessity.

Conversely from (3.12), (3.13) and (3.16), we have

$$G^i = \bar{G}^i + \left[\theta + \left(\frac{\alpha^3c_1c_2 - 3c_2\alpha\beta^2 - 4\beta^3}{c_1\alpha^2 + c_2\alpha\beta + \beta^2}\right)\tau + \frac{1}{\bar{b}^2}\left(\bar{s}_0 + \frac{\bar{r}_{00}\bar{\beta}}{\bar{\alpha}^2}\right)\right]y^i$$

Thus F is projectively equivalent to $\bar{F}$.

Hence the proof.

From the above theorem, (3.2) and (3.3), we have the following corollary;

Corrollary 3.1: Let $F = c_1\alpha + c_2\beta + \frac{\beta^2}{\alpha};\ c_2 \neq 0$ and $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ be two (α,β)-metrics on an n-dimensional manifold M with dimension $n \geq 3$, where α and $\bar{\alpha}$ are two Riemannian metrics, β and $\bar{\beta}$ are two non zero collinear 1-forms. Then F is projectively related to $\bar{F}$ if and only if the following are holds true,

$$G^i_\alpha + 2\alpha^2\tau b^i = \bar{G}^i_{\bar{\alpha}} + \frac{1}{2\bar{b}^2}\left(\bar{\alpha}^2\bar{s}^i + \bar{r}_{00}\bar{b}^i\right) + \theta y^i$$

$$\bar{s}_{ij} = \frac{1}{\bar{b}^2}\left(\bar{b}_i\bar{s}_j - \bar{b}_j\bar{s}_i\right)$$

$$b_{i|j} = 2\tau\left\{\left(1+\frac{2b^2}{c_1}\right)a_{ij} - \left(\frac{3}{c_1}\right)b_i b_j\right\},$$

where $b_{i|j}$ denote the coeeficient of the covariant derivatives of β with respect to $\alpha.$

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